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Slow convergence of sequences of linear operators II: Arbitrarily slow convergence

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Abstract

We study the *rate of convergence* of a sequence of linear operators that converges *pointwise* to a linear operator. Our main interest is in characterizing the slowest type of pointwise convergence possible. This is a continuation of the paper Deutsch and Hundal (2010) [14]. The main result is a "lethargy" theorem (Theorem 3.3) which gives useful conditions that guarantee arbitrarily slow convergence. In the particular case when the sequence of linear operators is generated by the powers of a single linear operator, we obtain a "dichotomy" theorem, which states the surprising result that either there is linear (fast) convergence or arbitrarily slow convergence; no other type of convergence is possible. The dichotomy theorem is applied to generalize and sharpen: (1) the von Neumann–Halperin cyclic projections theorem, (2) the rate of convergence for intermittently (i.e., "almost" randomly) ordered projections, and (3) a theorem of Xu and Zikatanov.

Keywords: Arbitrarily slow convergence; Higher powers of linear operators; Cyclic projections; Alternating projections; Randomly ordered projections; Intermittently ordered projections; Subspace corrections; Finite elements; Domain decomposition; Multigrid method; Rate of convergence

1. Introduction

There are some important algorithms in analysis that are all special cases of the following type. Let (L_n) be sequence of bounded linear operators from a Banach space X to a normed

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linear space Y, and suppose that the sequence converges *pointwise* to a bounded linear operator L, that is,

$$L(x) := \lim_{n\to\infty} L_n(x) \quad \text{for each } x \in X.$$

A natural and practical question that arises then is: What can be said about the *rate* or speed of this convergence? This is an interesting and important question that does not seem to have been studied in a general systematic way before. In the paper [14] we began such a theoretical study with the main emphasis on convergence that is "extremely slow", along with numerous applications. It is the object of this paper to continue this study by giving conditions that imply the *slowest* possible type of convergence, namely, "arbitrarily slow convergence", and to give more applications.

Recall that arbitrarily slow convergence was defined in [14]. (The sequence (L_n) is said to converge to L *arbitrarily slowly* provided it converges pointwise and, for each sequence of real numbers $(\phi(n))$ with $\lim_n \phi(n) = 0$, there exists $x = x_\phi \in X$ such that $\|L_n(x) - L(x)\| \geq \phi(n)$ for each n.) The phrase "arbitrarily slow convergence" has appeared in several papers. But in many of these, no precise definition was given. But even the precise definitions differed in a significant way. However, Schock [29] did give such a definition for a special class of methods for obtaining approximate solutions to a particular linear operator equation. In [14] we extended Schock's definition to a more general setting and showed that his definition was equivalent to what we called there "almost arbitrarily slow convergence" (see [14, Lemma 2.11]).

Section 3 contains the main result of the paper, a "lethargy" theorem (Theorem 3.3). It provides essential sufficient conditions guaranteeing that the sequence (L_n) converges to L arbitrarily slowly. Furthermore, Theorem 3.3 is the basis for all the main results and applications in Sections 4 and 6–8.

In Section 4 we consider the important special case when the sequence (L_n) is generated by the powers of a *single* linear operator T, i.e., $L_n = T^n$ for each n. The main result here is a "dichotomy" theorem (Theorem 4.4): If $\|T\| \leq 1$ and (T^n) converges pointwise to 0, then either $\|T^{n_1}\| < 1$ for some n_1 (in which case (T^n) converges to 0 linearly), or $\|T^n\| = 1$ for all n (in which case (T^n) converges to 0 arbitrarily slowly). This shows that, in the case of powers, there are exactly two different types of convergence possible: either linear (possibly finite) or arbitrarily slow. There are no intermediate types of pointwise convergence possible.

In Section 5 we compare our lethargy theorem with a classical "lethargy theorem" of Bernstein.

In Section 6 we apply Theorem 4.4 to sharpen and improve the von Neumann–Halperin theorem. Briefly, (Theorem 6.4) if $M_1, M_2, \ldots, M_r$ are closed subspaces in a Hilbert space X, then exactly one of two statements holds: either (1) $\sum_1^r M_i^\perp$ is closed (in which case $((P_{M_r} P_{M_{r-1}} \cdots P_{M_1})^n)$ converges to $P_{\cap_1^r M_i}$ linearly), or (2) $\sum_1^r M_i^\perp$ is not closed (in which case $((P_{M_r} P_{M_{r-1}} \cdots P_{M_1})^n)$ converges to $P_{\cap_1^r M_i}$ arbitrarily slowly). (Here P_M denotes the orthogonal projection onto M.) This generalizes a result established for the special case of two subspaces, where it was proved by a combination of the results of Bauschke et al. [5] and Bauschke et al. [7].

In Section 7 we obtain a generalization of Theorem 6.4 to the situation where the projections are *intermittently ordered*, which is "almost" randomly ordered (and not necessarily cyclically ordered).

In Section 8 we apply Theorem 4.4 to obtain a rate of convergence result that sharpens and improves one of the main results of Xu and Zikatanov [33].