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Convergence rates of approximate sums of Riemann integrals

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Abstract

We represent the convergence rates of the Riemann sums and the trapezoidal sums with respect to regular divisions and optimal divisions of a bounded closed interval to the Riemann integrals as some limits of their expanded error terms.

Keywords: Convergence rate; Riemann sum; Riemann integral; Trapezoidal sum

1. Introduction

The Riemann sums and the trapezoidal sums of functions defined on a bounded closed interval are well known as approximate sums of the Riemann integrals of the functions. In this paper the author represents the convergence rates of the Riemann sums and the trapezoidal sums as some limits of their expanded error terms.

Let $[a, b]$ be a bounded closed interval. We take an n-division Δ of $[a, b]$ defined by

$$\Delta : a = s_0 \le s_1 \le \cdots \le s_{n-1} \le s_n = b.$$

We denote by D_n the division of $[a, b]$ defined by $s_i = a + i(b - a)/n$ and call it the regular n-division. For a function f defined on $[a, b]$ and $s_{i-1} \le \xi_i \le s_i$ we define the Riemann sum $R(f; \Delta, \xi_i)$ by

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$$R(f; \Delta, \xi_i) = \sum_{i=1}^{n} (s_i - s_{i-1}) f(\xi_i).$$

The width $d(\Delta)$ of Δ is defined as $d(\Delta) = \max\{s_i - s_{i-1} \mid 1 \leq i \leq n\}$. The Riemann integral of f is defined as

$$\int_a^b f(x)\mathrm{d}x = \lim_{d(\Delta)\to 0} R(f; \Delta, \xi_i)$$

and textbooks on calculus usually show that this limit exists for a continuous function f. In this paper we consider some limits of expanded error terms like

$$n \left| \int_a^b f(x)\mathrm{d}x - R(f; \Delta, \xi_i) \right|, \qquad n^2 \left| \int_a^b f(x)\mathrm{d}x - R(f; \Delta, \xi_i) \right|$$

as $n \to \infty$. Chui [1] obtained such a limit of an expanded error term.

Theorem 1.1 (*Chui*). *If f is twice differentiable and f'' is bound and almost everywhere continuous on $[a, b]$, then*

$$\lim_{n\to\infty} n^2 \left\{ \int_a^b f(x)\mathrm{d}x - R\left(f; D_n, \frac{1}{2}(s_{i-1} + s_i)\right) \right\}$$
$$= \frac{(b-a)^2}{24} \int_a^b f''(x)\mathrm{d}x = \frac{(b-a)^2}{24} (f'(b) - f'(a)).$$

In [1] the above theorem is formulated for the interval [0, 1].

We consider not only regular divisions D_n but also optimal divisions for lower Riemann sums and trapezoidal sums, so we explain about optimal divisions. We take a continuous function f defined on $[a, b]$. For any division Δ of $[a, b]$ we take $s_{i-1} \leq \xi_i \leq s_i$ which satisfies $f(\xi_i) = \min_{[s_{i-1}, s_i]} f$ and define the lower Riemann sum

$$R(f; \Delta, \min) = R(f; \Delta, \xi_i).$$

The set of all n-divisions of $[a, b]$ is compact and

$$\Delta \mapsto R(f; \Delta, \min)$$

is continuous, so there exists an n-division $\Delta_n^\#$ at which the above function attains its maximum. This n-division $\Delta_n^\#$ is optimal for the lower Riemann sum $R(f; \Delta, \min)$. It may not be unique, but the sum $R(f; \Delta_n^\#, \min)$ is unique. Thus we can consider $R(f; \Delta_n^\#, \min)$. One of the main theorems of this paper is as follows:

Theorem 1.2. *If f is a function of class C^1 defined on $[a, b]$, then*

$$\lim_{n\to\infty} n \left\{ \int_a^b f(x)\mathrm{d}x - R(f; \Delta_n^\#, \min) \right\} = \frac{1}{2} \left(\int_a^b |f'(x)|^{1/2} \mathrm{d}x \right)^2.$$

The trapezoidal sum $T(f; \Delta)$ of f is defined as

$$T(f; \Delta) = \sum_{i=1}^{n} (s_i - s_{i-1}) \frac{1}{2} (f(s_{i-1}) + f(s_i)).$$

We can obtain the limit of the expanded error term of the trapezoidal sum as follows: