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Convexities and approximative compactness and continuity of metric projection in Banach spaces[☆]

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Abstract

In this paper, we investigate the continuities of the metric projection in a nonreflexive Banach space X, which improve the results in [X.N. Fang, J.H. Wang, Convexity and continuity of metric projection, Math. Appl. 14 (1) (2001) 47–51; P.D. Liu, Y.L. Hou, A convergence theorem of martingales in the limit, Northeast. Math. J. 6 (2) (1990) 227–234; H.J. Wang, Some results on the continuity of metric projections, Math. Appl. 8 (1) (1995) 80–84]. Under the assumption that X has some convexities, we discuss the relationship between approximative compactness of a subset A of X and continuity of the metric projection P_A. We also give a representation theorem for the metric projection to a hyperplane in dual space X^* and discuss its continuity.

Keywords: Metric projection; Proximinal set; Approximative compactness; Upper semi-continuity; Convexity

1. Introduction

Let X and X^* be a Banach space and its dual space, $S(X)$ and $B(X)$ be the unit sphere and unit ball of X, respectively. Let $x \in S(X)$, $\Sigma(x) = \{x^* \in S(X^*) : x^*(x) = 1\}$. For $A \subseteq X$, the

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metric projection $P_A : X \to 2^A$ is defined by $P_A(x) = \{y \in A :\| x - y \|= d(x, A)\}$, where $d(x, A) = \inf\{\|x - y\| : y \in A\}$, $x \in X$. If $P_A(x) \neq \emptyset$ for each $x \in X$, then A is said to be *proximinal*. If $P_A(x)$ is at most a singleton for each $x \in X$, then A is said to be *semi-Chebyshev*. If A is simultaneously a *proximinal* and *semi-Chebyshev* set, then A is called the *Chebyshev* set. For any $\varepsilon > 0$, an element z in A is called a ε-approximator for x if $\| x - z \|\leq d(x, A) + \varepsilon$, the set of all ε-approximator of x in A is denoted by $P_A^\varepsilon(x)$.

Let $\{A_n\}$ be a sequence of nonempty subsets of a Banach space X. Define

$$w - \overline{\lim_n} A_n = \{x \in X : \exists x_{n_k} \in A_{n_k}, x = w - \lim_k x_{n_k}\}$$

and

$$s - \underline{\lim_n} A_n = \{x \in X : \exists x_n \in A_n, x = \lim_n x_n\}.$$

A sequence of nonempty subsets $\{A_n\}$ of a Banach space X is said to converge to a set A in the Mosco sense if $s - \underline{\lim_n} A_n = w - \overline{\lim_n} A_n = A$, which is denoted by $\lim_{nM} A_n = A$. The sequence $\{A_n\}$ is said to converge to a set A in the Wijsman sense if $\lim_n d(x, A_n) = d(x, A)$ for all x in X, which is denoted by $\lim_{nW} A_n = A$. The sequence $\{A_n\}$ is said to converge to a set A in the strong Wijsman sense if for all x in X and all $\varepsilon_n \to 0^+$ as $n \to \infty$, we have $\lim_n \|x - u_n\| = d(x, A)$ where $u_n \in co(\cup_{k=n}^\infty P_{A_k}^{\varepsilon_k}(x))$ for $n \geq 1$, which is denoted by $s - \lim_{nW} A_n = A$. By [13], we know that $s - \lim_{nW} A_n = A$ implies $\lim_{nW} A_n = A$, and $\lim_{nM} A_n = A$ implies $s - \lim_{nW} A_n = A$ in reflexive Banach spaces.

The mapping $x \to P_A(x)$ is said to be (weakly) upper semi-continuous if for all x in X and for all (weakly) open set $W \supset P_A(x)$, there exists a neighborhood U of x such that $P_A(U) \subset W$. The mapping $P : (x, A) \to P_A(x)$ is said to be (weakly) upper semi-continuous at (x, A) in the *strong Wijsman–Zhang sense* if for all (weakly) open set $W \supset P_A(x)$ and for all $x_n \to x$ and all *proximinal* set sequence $\{A_n \cap A\}$ with $s - \lim_{nW}(A_n \cap A) = A$, there exists a positive integer N such that $P_{A_n \cap A}(x_n) \subset W$ for all $n \geq N$. The mapping $P : (x, A) \to P_A(x)$ is called (weakly) upper semi-continuous in the *strong Wijsman–Zhang sense* if it is so at all (x, A).

A nonempty subset A of X is said to be (weakly) *approximatively compact* if any sequences $\{x_n\}_{n\in\mathbb{N}} \subset A$ and any $y \in X$ such that $\lim_n \|x_n - y\| = d(y, A)$, have a (weakly convergent subsequence) Cauchy subsequence. X is called (weakly) *approximatively compact* if any nonempty closed and convex subset of X is (weakly) *approximatively compact*.

A Banach space X is said to have the property (C-I) (resp.(C-II) or (C-III)) if for $x \in S(X)$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq B(X)$ such that for each $\delta > 0$ there exists a positive integer $N(\delta)$ satisfying

$$co(\{x\} \cup \{x_n : n \geq N(\delta)\}) \cap (1 - \delta)B(X) = \emptyset,$$

we have $\lim_n \| x_n - x \|= 0$ (resp. $\{x_n\}_{n\in\mathbb{N}}$ is relatively compact or $\{x_n\}_{n\in\mathbb{N}}$ is relatively weakly compact).

A Banach space X is said to be *strongly convex (resp. very convex/nearly strongly convex/nearly very convex)* [11,18] if for any $x \in S(X)$ and $\{x_n\}_{n\in\mathbb{N}} \subset B(X)$ with that $x^*(x_n) \to 1$ for some $x^* \in \Sigma(x)$, then $\{x_n\}_{n\in\mathbb{N}}$ is convergent (resp. weakly convergent/relatively compact/relatively weakly compact). By [18], we know that property (C-I) (resp. (C-II)/(C-III)) implies strong convexity (resp. nearly strong convexity/nearly very convexity).

In 1989, Borwein and Fitzpatrick [1] studied the relationship between Mosco convergence and the Kadec property. In 1980, Brosowski, Deutsch and Neürnberge [2] first considered the family $\{A_t\}_{t\in T}$ of subsets in a normed space X parameterized by a topological space T and discussed the continuity of the mapping $t \to P_{A_t}(x)$. In 1984, Tsukada [12] proved that in a