Available online at www.sciencedirect.com
ScienceDirect

Journal of Approximation Theory 162 (2010) 1322–1346

Journal of Approximation Theory
www.elsevier.com/locate/jat

The linear pencil approach to rational interpolation

Bernhard Beckermann[a,*], Maxim Derevyagin[a,b], Alexei Zhedanov[c]

[a] *Laboratoire Painlevé UMR 8524 (ANO-EDP), UFR Mathématiques – M3, UST Lille, F-59655 Villeneuve d'Ascq Cedex, France*
[b] *Department of Nonlinear Analysis, Institute of Applied Mathematics and Mechanics, R. Luxemburg Street 74, 83114 Donetsk, Ukraine*
[c] *Institute for Physics and Engineering, R. Luxemburg Street 72, 83114 Donetsk, Ukraine*

Received 22 August 2009; received in revised form 3 February 2010; accepted 11 February 2010
Available online 18 February 2010

Communicated by Guillermo López Lagomasino

Abstract

It is possible to generalize the fruitful interaction between (real or complex) Jacobi matrices, orthogonal polynomials and Padé approximants at infinity by considering rational interpolants, (bi)orthogonal rational functions and linear pencils $zB - A$ of two tridiagonal matrices A, B, following Spiridonov and Zhedanov.

In the present paper, as well as revisiting the underlying generalized Favard theorem, we suggest a new criterion for the resolvent set of this linear pencil in terms of the underlying associated rational functions. This enables us to generalize several convergence results for Padé approximants in terms of complex Jacobi matrices to the more general case of convergence of rational interpolants in terms of the linear pencil. We also study generalizations of the Darboux transformations and the link to biorthogonal rational functions. Finally, for a Markov function and for pairwise conjugate interpolation points tending to ∞, we compute the spectrum and the numerical range of the underlying linear pencil explicitly.

Keywords: Multipoint Padé approximation; Rational interpolation; MP continued fractions; Jacobi matrix; Linear pencils

1. Introduction

The connection with Jacobi matrices has led to numerous applications of spectral techniques for self-adjoint operators in the theory of orthogonal polynomials on the real line and Padé

* Corresponding author.
E-mail addresses: bbecker@math.univ-lille1.fr (B. Beckermann), derevyagin.m@gmail.com (M. Derevyagin), zhedanov@yahoo.com (A. Zhedanov).

0021-9045/$ - see front matter
doi:10.1016/j.jat.2010.02.004

approximation. In order to give an idea of these interactions consider a Markov function of the form

$$\varphi(z) = \int_a^b \frac{\mathrm{d}\mu(t)}{z-t},$$

where a, b are real numbers and $\mathrm{d}\mu(t)$ is a probability measure; that is, $\int_a^b \mathrm{d}\mu(t) = 1$. It is well known [1,32] that one can expand such a Markov function φ into the following continued fraction:

$$\varphi(z) = \cfrac{1}{z - b_0 - \cfrac{a_0^2}{z-b_1-\cfrac{a_1^2}{\ddots}}} = \frac{1 |}{| z - b_0} - \frac{a_0^2 |}{| z - b_1} - \frac{a_1^2 |}{| z - b_2} - \cdots, \tag{1.1}$$

where $b_j, a_j \in \mathbb{R}$, $a_j > 0$. Continued fractions of the form (1.1) are called J-fractions [23,32]. To the continued fraction (1.1) one can associate a Jacobi matrix A acting in the space of square summable sequences and its truncation $A_{[0:n]}$:

$$A = \begin{pmatrix} b_0 & a_0 & & \\ a_0 & b_1 & a_1 & \\ & a_1 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \qquad A_{[0:n-1]} = \begin{pmatrix} b_0 & a_0 & & \\ a_0 & b_1 & \ddots & \\ & \ddots & \ddots & a_{n-2} \\ & & a_{n-2} & b_{n-1} \end{pmatrix}.$$

Then it is known that $\varphi(z) = \langle (zI - A)^{-1} e_0, e_0 \rangle$, and the nth convergent of the above continued fraction is given by

$$\frac{p_n(z)}{q_n(z)} = \langle (zI - A_{[0:n-1]})^{-1} e_0, e_0 \rangle = \frac{1 |}{| z - a_0} - \cdots - \frac{b_{n-2}^2 |}{| z - a_{n-1}},$$

where the column vector $e_0 = (1, 0, \ldots)^\top$ is the first canonical vector of suitable size, q_n are orthogonal polynomials with respect to $\mathrm{d}\mu$, and p_n are polynomials of the second kind; see [1,25,26]. It is an elementary fact of continued fraction theory that

$$\varphi(z) - \frac{p_n(z)}{q_n(z)} = O\left(\frac{1}{z^{2n+1}}\right)_{z\to\infty}; \tag{1.2}$$

see for instance [1,4,23]. Relation (1.2) means that the rational function p_n/q_n is the nth diagonal Padé approximant to φ at infinity. Consequently, the locally uniform convergence of diagonal Padé approximants appears as the strong resolvent convergence of the finite matrix approximations $A_{[0:n]}$. For instance, one knows that $p_n/q_n \to \varphi$ in capacity in the resolvent set of A given by the complement of the support of μ, and locally uniformly outside the numerical range of A given by the convex hull of the spectrum of A; see for instance [29]. In addition, it should be mentioned here that an operator approach for proving convergence of Padé approximants for rational perturbations of Markov functions was proposed in [17]; see also [16].

If φ is no longer a Markov function but has distinct nth diagonal Padé approximants at infinity, we may still recover these approximants as convergents of a continued fraction of type (1.1), but now in general $a_j, b_j \in \mathbb{C}$, $a_j \neq 0$; see [32]. Thus, A becomes complex symmetric, and is called a complex Jacobi matrix. There is no longer a natural candidate for the spectrum of A,