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Increasing the polynomial reproduction of a quasi-interpolation operator

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Abstract

Quasi-interpolation is an important tool, used both in theory and in practice, for the approximation of smooth functions from univariate or multivariate spaces which contain $\Pi_m = \Pi_m(\mathbb{R}^d)$, the d-variate polynomials of degree $\leq m$. In particular, the reproduction of Π_m leads to an approximation order of $m+1$. Prominent examples include Lagrange and Bernstein type approximations by polynomials, the orthogonal projection onto Π_m for some inner product, finite element methods of precision m, and multivariate spline approximations based on macroelements or the translates of a single spline.

For such a quasi-interpolation operator L which reproduces $\Pi_m(\mathbb{R}^d)$ and any $r \geq 0$, we give an explicit construction of a quasi-interpolant $R_m^{r+m} L = L + A$ which reproduces Π_{m+r}, together with an integral error formula which involves only the $(m+r+1)$th derivative of the function approximated. The operator $R_m^{m+r} L$ is defined on functions with r additional orders of smoothness than those on which L is defined. This very general construction holds in all dimensions d. A number of representative examples are considered.

Keywords: Quasi-interpolation; Lagrange interpolation; Bernstein polynomial; Finite element method; Multivariate polynomial approximation; Error formula; Multipoint Taylor formula; Divided differences; Chu–Vandermonde convolution

1. Introduction

A quasi-interpolant for a space F of approximating functions is a linear map L onto F which is bounded (in some relevant norm), local, and reproduces some polynomial space, see, e.g., [8]. When F is a univariate or multivariate space of polynomials or splines, quasi-interpolants provide useful approximations of smooth functions. These have both practical and theoretical advantages,

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e.g., the reproduction of the space $\Pi_m = \Pi_m(\mathbb{R}^d)$ of d-variate polynomials of degree $\le m$ leads to an approximation order of $m+1$. Some well known examples include Lagrange and Bernstein type approximations by polynomials, the orthogonal projection onto Π_m for some inner product, finite element methods of precision m, and multivariate spline approximations based on macroelements or the translates of a single spline.

The main result of this paper is the following. For any quasi-interpolant L which reproduces $\Pi_m(\mathbb{R}^d)$ and $r \ge 0$, we explicitly construct a quasi-interpolant

$$R_m^{r+m} L = L + A$$

which reproduces Π_{m+r}, together with an integral error formula $E(f,x)$, which involves only the $(m+r+1)$th derivative of the function approximated. The quasi-interpolant $R_m^{m+r} L$ allows the order of approximation by L to be increased by r, with the trade off being that it is defined on functions with r additional orders of smoothness than those on which L is defined. The operation $L \mapsto R_m^{m+r} L$ has many nice properties, including being defined for all dimensions d, being continuous (in an appropriate sense), and satisfying

$$R_{m+r_1}^{m+r_1+r_2} R_m^{m+r_1} L = R_m^{m+r_1+r_2} L. \tag{1.1}$$

The error formula for approximation by R_m^{r+m} can be interpreted as an asymptotic expansion of the error in approximation by L, i.e.,

$$f(x) - Lf(x) = Af(x) + E(f,x).$$

By way of comparison, the Voronovskaya type asymptotic expansion for the Bernstein operator (see [5, Section 1.6.1], or [6]) involves the derivatives of f at x, and hence the corresponding function $x \mapsto Af(x)$ is not a polynomial, while in our case it is (see [4, Section 3.1]).

The paper is set out as follows. In the remainder of this section, we give precise definitions and establish notation. Next we give a multivariate divided difference involving two points upon which our results are based. The following section then uses this to prove the main result, and gives some representative examples. The final section establishes the remarkable formula (1.1), which requires some technical calculations.

1.1. Basic definitions and notation

The (directional) derivative of a function f in the direction $v \in \mathbb{R}^d$ at a point $x \in \mathbb{R}^d$ is denoted by

$$D_v f(x) := \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}.$$

We note that $v \mapsto D_v f(x)$ is linear. In particular, for the univariate case $d = 1$ one has

$$D_{x-y}^k f = (x-y)^k f^{(k)}, \qquad x, y \in \mathbb{R}, \tag{1.2}$$

where $f^{(k)}$ denotes the k-th derivative of a univariate function, and $D_v^k := (D_v)^k$. Let $D_j := D_{e_j}$, where e_j is the j-th standard basis vector in $\mathbb{R}^d$. Then the α-th partial derivative $D^\alpha f$ of a function f with a k-th derivative is

$$D^\alpha f := D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} f, \qquad |\alpha| := \alpha_1 + \cdots + \alpha_n = k.$$

We call $D^k f := (D^\alpha f)_{|\alpha|=k}$ the k-th derivative of f.