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# Bregman distances and Klee sets

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**Abstract**

In 1960, Klee showed that a subset of a Euclidean space must be a singleton provided that each point in the space has a unique farthest point in the set. This classical result has received much attention; in fact, the Hilbert space version is a famous open problem. In this paper, we consider Klee sets from a new perspective. Rather than measuring distance induced by a norm, we focus on the case when distance is meant in the sense of Bregman, i.e., induced by a convex function. When the convex function has sufficiently nice properties, then – analogously to the Euclidean distance case – every Klee set must be a singleton. We provide two proofs of this result, based on Monotone Operator Theory and on Nonsmooth Analysis. The latter approach leads to results that complement the work by Hiriart-Urruty on the Euclidean case.


*Keywords:* Convex function; Legendre function; Bregman distance; Bregman projection; Farthest point; Maximal monotone operator; Subdifferential operator

## 1. Introduction

Throughout this paper, $\mathbb{R}^J$ denotes the standard Euclidean space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $C$ be a nonempty bounded closed subset of $\mathbb{R}^J$ and assume that $C$ is

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a *Klee set (with respect to the Euclidean distance)*, i.e., each point in $\mathbb{R}^J$ has a unique farthest point in $C$. Must $C$ be a singleton? The *farthest-point conjecture* [11] states that the answer to this question is affirmative. This conjecture has attracted many mathematicians; see, e.g., [4,10–13,25] and the references therein. Although the farthest-point conjecture is true in $\mathbb{R}^J$, as was shown originally by Klee [14] (see also [1,11,17]), only partial results are known in infinite-dimensional settings (see, e.g., [18,25]). While the farthest-point conjecture is primarily of theoretical interest, it should be noted that farthest points do play a role in computational geometry; see, e.g., the sections on Voronoi diagrams in [20].

In this paper, we cast a new light on this problem by measuring the distance in the sense of Bregman rather than in the usual Euclidean sense. To this end, assume that

$$f : \mathbb{R}^J \to \left]-\infty, +\infty\right] \text{ is convex and differentiable on } U := \operatorname{int}\operatorname{dom} f \neq \varnothing, \tag{1}$$

where $\operatorname{int}\operatorname{dom} f$ stands for the interior of the set $\operatorname{dom} f := \{x \in \mathbb{R}^J \mid f(x) \in \mathbb{R}\}$. Then the *Bregman distance* [5] with respect to $f$, written as $D_f$ or simply $D$, is

$$D : \mathbb{R}^J \times \mathbb{R}^J \to [0, +\infty] : (x, y) \mapsto \begin{cases} f(x) - f(y) - \langle \nabla f(y), x - y \rangle, & \text{if } y \in U; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

Although standard, it is well known that the name "Bregman distance" is somewhat misleading because in general $D$ is neither symmetric nor does the triangle inequality hold. We recommend books [6,7] to the reader for further information on Bregman distances and their various applications.

Throughout, we assume that

$$C \subset U. \tag{3}$$

Now define the *left-Bregman-farthest-distance function* by

$$\overleftarrow{F}_C : U \to [0, +\infty] : y \mapsto \sup_{x \in C} D(x, y), \tag{4}$$

and the corresponding *left-Bregman-farthest-point map* by

$$\overleftarrow{Q}_C : U \to U : y \mapsto \operatorname*{argmax}_{x \in C} D(x, y). \tag{5}$$

Since $D$ is in general not symmetric, there exist analogously the *right-Bregman-farthest-distance function* and the *right-Bregman-farthest-point map*. These objects, which we will study later, are denoted by $\overrightarrow{F}_C$ and $\overrightarrow{Q}_C$, respectively. When $f = \frac{1}{2}\|\cdot\|^2$, then $D : (x, y) \mapsto \frac{1}{2}\|x - y\|^2$ is symmetric and the corresponding map $\overleftarrow{Q}_C$ is identical to the farthest-point map with respect to the Euclidean distance.

The present more general framework based on Bregman distances allows for significant extensions of Hiriart-Urruty's work [11] (and for variants of some of the results in [25]). One of our main results states that if $f$ is sufficiently nice, then every Klee set (with respect to $D$) must be a singleton. Two fairly distinct proofs of this result are given. The first is based on the deep Brézis–Haraux range approximation theorem from monotone operator theory. The second proof, which uses generalized subdifferentials from nonsmooth analysis, allows us to characterize sets with unique farthest points. Various subdifferentiability properties of the Bregman-farthest-distance function are also provided. The present work complements a corresponding study on Chebyshev sets [3], where the focus is on *nearest* rather than farthest points.