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Reverse triangle inequalities for potentials

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Abstract

We study the reverse triangle inequalities for suprema of logarithmic potentials on compact sets of the plane. This research is motivated by the inequalities for products of supremum norms of polynomials. We find sharp additive constants in the inequalities for potentials, and give applications of our results to the generalized polynomials.

We also obtain sharp inequalities for products of norms of the weighted polynomials $w^n P_n$, $\deg(P_n) \le n$, and for sums of potentials with external fields. An important part of our work in the weighted case is a Riesz decomposition for the weighted farthest-point distance function.

Keywords: Potentials; Polynomials; Supremum norm; Logarithmic capacity; Equilibrium measure; Subharmonic function; Fekete points

1. Products of polynomials and sums of potentials

Let E be a compact set in the complex plane $\mathbb{C}$. Given the bounded above functions f_j, $j = 1, \ldots, m$, on E, we have by a standard inequality that

$$\sup_E \sum_{j=1}^m f_j \le \sum_{j=1}^m \sup_E f_j.$$

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It is not possible to reverse this inequality for arbitrary functions, even if one introduces additive or multiplicative "correction" constants. However, we are able to prove the reverse inequalities for *logarithmic potentials*, with sharp additive constants. For a positive Borel measure μ with compact support in the plane, define its (subharmonic) potential [1, p. 53] by

$$p(z) := \int \log|z-t|\mathrm{d}\mu(t).$$

Let ν_j, $j = 1, \ldots, m$, be positive compactly supported Borel measures with potentials p_j. We normalize the problem by assuming that the total mass of $\nu := \sum_{j=1}^m \nu_j$ is equal to 1, and consider the inequality

$$\sum_{j=1}^m \sup_E p_j \le C_E(m) + \sup_E \sum_{j=1}^m p_j. \tag{1.1}$$

Clearly, if (1.1) holds true, then $C_E(m) \ge 0$. One may also ask whether (1.1) holds with a constant C_E independent of m. The motivation for such inequalities comes directly from the inequalities for the norms of products of polynomials. Indeed, if $P(z) = \prod_{j=1}^n (z - a_j)$ is a monic polynomial, then $\log|P(z)| = n \int \log|z-t|\mathrm{d}\tau(t)$. Here, $\tau = \frac{1}{n}\sum_{j=1}^n \delta_{a_j}$ is the normalized counting measure in the zeros of P, with δ_{a_j} being the unit point mass at a_j. Let $\|P\|_E := \sup_E |P|$ be the uniform (sup) norm on E. Thus (1.1) takes the following form for polynomials P_j, $j = 1, \ldots, m$,

$$\prod_{j=1}^m \|P_j\|_E \le \mathrm{e}^{nC_E(m)} \left\| \prod_{j=1}^m P_j \right\|_E,$$

where n is the degree of the product polynomial $\prod_{j=1}^m P_j$. We outline a brief history of such inequalities below.

Kneser [2] proved the first sharp inequality for norms of products of polynomials on $[-1, 1]$ (see also Aumann [3] for a weaker result)

$$\|P_1\|_{[-1,1]}\|P_2\|_{[-1,1]} \le K_{\ell,n}\|P_1P_2\|_{[-1,1]}, \quad \deg P_1 = \ell,\ \deg P_2 = n-\ell, \tag{1.2}$$

where

$$K_{\ell,n} := 2^{n-1}\prod_{k=1}^{\ell}\left(1+\cos\frac{2k-1}{2n}\pi\right)\prod_{k=1}^{n-\ell}\left(1+\cos\frac{2k-1}{2n}\pi\right). \tag{1.3}$$

Observe that equality holds in (1.2) for the Chebyshev polynomial $t(x) = \cos n \arccos x = P_1(x)P_2(x)$, with a proper choice of the factors $P_1(x)$ and $P_2(x)$. Borwein [4] generalized this to the multifactor inequality

$$\prod_{j=1}^m \|P_j\|_{[-1,1]} \le 2^{n-1}\prod_{k=1}^{[\frac{n}{2}]}\left(1+\cos\frac{2k-1}{2n}\pi\right)^2 \left\|\prod_{j=1}^m P_j\right\|_{[-1,1]}, \tag{1.4}$$

where n is the degree of $\prod_{j=1}^m P_j$. We remark that

$$2^{n-1}\prod_{k=1}^{[\frac{n}{2}]}\left(1+\cos\frac{2k-1}{2n}\pi\right)^2 \sim (3.20991\ldots)^n \quad \text{as } n \to \infty. \tag{1.5}$$