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On the sampling and recovery of bandlimited functions via scattered translates of the Gaussian

Th. Schlumprecht*, N. Sivakumar

Department of Mathematics, Texas A&M University College Station, TX 77843, USA

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Abstract

Let λ be a positive number, and let $(x_j : j \in \mathbb{Z}) \subset \mathbb{R}$ be a fixed Riesz-basis sequence, namely, (x_j) is strictly increasing, and the set of functions $\{\mathbb{R} \ni t \mapsto e^{ix_j t} : j \in \mathbb{Z}\}$ is a Riesz basis (*i.e.,* unconditional basis) for $L_2[-\pi, \pi]$. Given a function $f \in L_2(\mathbb{R})$ whose Fourier transform is zero almost everywhere outside the interval $[-\pi, \pi]$, there is a unique sequence $(a_j : j \in \mathbb{Z})$ in $\ell_2(\mathbb{Z})$, depending on λ and f, such that the function

$$I_\lambda(f)(x) := \sum_{j \in \mathbb{Z}} a_j e^{-\lambda(x-x_j)^2}, \quad x \in \mathbb{R},$$

is continuous and square integrable on $(-\infty, \infty)$, and satisfies the interpolatory conditions $I_\lambda(f)(x_j) = f(x_j)$, $j \in \mathbb{Z}$. It is shown that $I_\lambda(f)$ converges to f in $L_2(\mathbb{R})$, and also uniformly on $\mathbb{R}$, as $\lambda \to 0^+$. In addition, the fundamental functions for the univariate interpolation process are defined, and some of their basic properties, including their exponential decay for large argument, are established. It is further shown that the associated interpolation operators are bounded on $\ell_p(\mathbb{Z})$ for every $p \in [1, \infty]$.

Keywords: Scattered data; Gaussian interpolation; Bandlimited functions

1. Introduction

This paper, one in the long tradition of those involving the interpolatory theory of functions, is concerned with interpolation of data via the translates of a Gaussian kernel. The motivation

* Corresponding author.
E-mail addresses: thomas.schlumprecht@math.tamu.edu, schlump@math.tamu.edu (Th. Schlumprecht), sivan@math.tamu.edu (N. Sivakumar).

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for this work is twofold. The first is the theory of *Cardinal Interpolation*, which deals with the interpolation of data prescribed at the integer lattice, by means of the integer shifts of a single function. This subject has a rather long history, and it enjoys interesting connections with other branches of pure and applied mathematics, *e.g.* Toeplitz matrices, Function Theory, Harmonic Analysis, Sampling Theory. When the underlying function (whose shifts form the basis for interpolation) is taken to be the so-called *Cardinal B-Spline*, one deals with *Cardinal Spline Interpolation*, a subject championed by Schoenberg, and taken up in earnest by a host of followers. More recently, it was discovered that there is a remarkable analogy between cardinal spline interpolation and cardinal interpolation by means of the (integer) shifts of a Gaussian, a survey of which may be found in [1]. The current article may also be viewed as a contribution in this vein; it too explores further connections between the interpolatory theory of splines and that of the Gauss kernel, but does so in the context of interpolation at point sets which are more general than the integer lattice. This brings us to the second, and principal, motivating influence for our work, namely the researches of Lyubarskii and Madych [2]. This duo have considered spline interpolation at certain sets of points which are generalizations of the integer lattice, and we were prompted by their work to ask if the analogy between splines and Gaussians, very much in evidence in the context of cardinal interpolation, persists in this 'non-uniform' setting also. Our paper seeks to show that this is indeed the case. The influence of [2] on our work goes further. Besides providing us with the motivating question for our studies, it also offered us an array of basic tools which we have modified and adapted.

We shall supply more particulars – of a technical nature – concerning the present paper later in this introductory section, soon after we finish discussing some requisite general material.

Throughout this paper $L_p(\mathbb{R})$ and $L_p[a, b]$, $1 \leq p \leq \infty$, will denote the usual Lebesgue spaces over $\mathbb{R}$ and the interval $[a, b]$, respectively. We shall let $C(\mathbb{R})$ be the space of continuous functions on $\mathbb{R}$, and $C_0(\mathbb{R})$ will denote the space of $f \in C(\mathbb{R})$ for which $\lim_{x\to\pm\infty} f(x) = 0$. An important tool in our analysis is the *Fourier Transform*, so we assemble some of its basic facts; our sources for this material are [3,4]. If $g \in L_1(\mathbb{R})$, then the Fourier transform of g, $\widehat{g}$, is defined as follows:

$$\widehat{g}(x) := \int_{-\infty}^{\infty} g(t)\mathrm{e}^{-\mathrm{i}xt}\mathrm{d}t, \quad x \in \mathbb{R}. \tag{1}$$

The Fourier transform of a $g \in L_2(\mathbb{R})$ will be denoted by $\mathcal{F}[g]$. It is known that $\mathcal{F}$ is a linear isomorphism on $L_2(\mathbb{R})$, and that the following hold:

$$\|\mathcal{F}[g]\|^2_{L_2(\mathbb{R})} = 2\pi \|g\|^2_{L_2(\mathbb{R})}, \quad g \in L_2(\mathbb{R}); \qquad \mathcal{F}[g] = \widehat{g}, \quad g \in L_2(\mathbb{R}) \cap L_1(\mathbb{R}). \tag{2}$$

Moreover, if $g \in L_2(\mathbb{R}) \cap C(\mathbb{R})$ and $\mathcal{F}[g] \in L_1(\mathbb{R})$, then the following inversion formula holds:

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}[g](x)\mathrm{e}^{\mathrm{i}xt}\mathrm{d}x, \quad t \in \mathbb{R}. \tag{3}$$

The functions we seek to interpolate are the so-called *bandlimited* or *Paley–Wiener functions*. Specifically, we define

$$PW_\pi := \left\{g \in L_2(\mathbb{R}) \,:\, \mathcal{F}[g] = 0 \text{ almost everywhere outside } [-\pi, \pi]\right\}.$$

Let $g \in PW_\pi$. The Fourier inversion formula implies

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}[g](x)\mathrm{e}^{\mathrm{i}xt}\mathrm{d}x = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{F}[g](x)\mathrm{e}^{\mathrm{i}xt}\mathrm{d}x, \tag{4}$$