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Regularity and the Cesàro–Nevai class

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Abstract

We consider OPRL and OPUC with measures regular in the sense of Ullman–Stahl–Totik and prove consequences on the Jacobi parameters or Verblunsky coefficients. For example, regularity on $[-2, 2]$ implies $\lim_{N\to\infty} N^{-1}[\sum_{n=1}^{N}(a_n - 1)^2 + b_n^2] = 0$.

Keywords: Orthogonal polynomial; Regular measure

1. Introduction and background

This paper concerns the general theory of orthogonal polynomials on the real line, OPRL (see [26,1,8,23]), and the unit circle, OPUC (see [26,9,18,19]). Ullman [27] introduced the notion of regular measure on $[-2, 2]$ (he used $[-1, 1]$; we use the normalization more common in the spectral theory literature): a measure, $d\mu$, on $\mathbb{R}$ with

$$\operatorname{supp}(d\mu) = [-2, 2] \tag{1.1}$$

and ($\{a_n, b_n\}_{n=1}^{\infty}$ are the Jacobi parameters of $d\mu$)

$$\lim_{n\to\infty} (a_1 \dots a_n)^{1/n} = 1. \tag{1.2}$$

Here we will look at the larger class with (1.1) replaced by

$$\sigma_{\text{ess}}(d\mu) = [-2, 2] \tag{1.3}$$

(i.e., $\operatorname{supp}(d\mu)$ is $[-2, 2]$ plus a countable set whose only limit points are a subset of $\{\pm 2\}$).

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Our goal is to explore what restrictions regularity places on the Jacobi parameters. At first sight, one might think (1.2) is the only restriction but, in fact, the combination of both (1.2) and (1.3) is quite strong. This should not be unexpected. After all, it is well known (going back at least to Nevai [15]; see also [19, Sect. 13.3]) that (1.1) plus $\liminf(a_1 \dots a_n) > 0$ imply

$$\sum_{n=1}^{\infty}(a_n - 1)^2 + b_n^2 < \infty. \tag{1.4}$$

One can use variational principles to deduce some restrictions on the a's and b's. For example, picking φ_n to be the vector in $\ell^2(\{1, 2, \dots\})$

$$\varphi_{n,j} = \begin{cases} \frac{1}{\sqrt{n}} & j \le n \\ 0 & j \ge n+1 \end{cases} \tag{1.5}$$

and using the Jacobi matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.6}$$

one sees, for example, that (1.3) implies (see also Theorem 1.2)

$$b_n \equiv 0 \Rightarrow \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} a_j \le 1 \tag{1.7}$$

$$a_n \equiv 1 \Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} b_j = 0. \tag{1.8}$$

In fact, we will prove much more:

Theorem 1.1. *If μ obeys (1.3) and (1.2), then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (|a_j - 1| + |b_j|) = 0. \tag{1.9}$$

Following the terminology for the OPUC analog of this in Golinskii–Khrushchev [10], we call (1.9) the Cesàro–Nevai condition and $\{a_j, b_j\}_{j=1}^{\infty}$ obeying (1.9) the Cesàro–Nevai class. It, of course, contains the Nevai class (named after [15]) where $|a_j - 1| + |b_j| \to 0$.

Noting that $\operatorname{supp}(d\mu)$ bounded implies

$$A = \sup_n (|a_n - 1| + |b_n|) < \infty \tag{1.10}$$

and that, by the Schwarz inequality,

$$\begin{aligned} \left(\frac{1}{n}\sum_{j=1}^{n} |a_j - 1| + |b_j|\right)^2 &\le \frac{2}{n}\sum_{j=1}^{n}(a_j - 1)^2 + (b_j)^2 \\ &\le 2A\,\frac{1}{n}\sum_{j=1}^{n}(|a_j - 1| + |b_j|) \end{aligned} \tag{1.11}$$