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On approximation properties of a family of linear operators at critical value of parameter

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Abstract

We introduce the family of linear operators

$$\left(A^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} \left(S_t f\right)(x)\, dt, \quad \alpha > 0$$

associated to a certain "admissible bunch" of operators S_t, $t > 0$, acting on $L_p(\mathbb{R}^n, dm$, and investigate the approximation properties of this family as $\alpha \to 0^+$. We give some applications to the Riesz and the Bessel potentials generated by the ordinary (Euclidean) and generalized translations.

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2 In translations from Russian, A.D. Gadjiev, A.D. Gadzhiev, and A.D. Gadžiev all refer to the same person.

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1. Introduction

In this paper, given some "admissible bunch" of linear operators $\{S_t\}_{t>0}$, acting on $L_p(\mathbb{R}^n, dm)$, we introduce the following family of integral operators:

$$(A^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (S_t f)(x)\, dt, \quad \alpha > 0.$$

This family of operators contains (for special choices of "admissible bunch" $\{S_t\}_{t>0}$) the Riesz and the Bessel potentials generated by the ordinary and generalized translation. The classical Riesz potentials, $I^\alpha f$, and the generalized Riesz potentials, $I_\nu^\alpha f$, are defined in terms of Fourier and Fourier–Bessel transforms by the following formulas:

$$F\left(I^\alpha f\right)(x) = |x|^{-\alpha} F(f)(x), \quad x \in \mathbb{R}^n; \tag{1}$$

$$F_\nu\left(I_\nu^\alpha f\right)(x) = |x|^{-\alpha} F_\nu(f)(x), \quad x \in \mathbb{R}_+^n. \tag{2}$$

Similarly, the classical Bessel potentials, $J^\alpha f$, and the generalized Bessel potentials, $J_\nu^\alpha f$, are defined as

$$F\left(J^\alpha f\right)(x) = \left(1 + |x|^2\right)^{-\alpha/2} F(f)(x), \quad x \in \mathbb{R}^n; \tag{3}$$

$$F_\nu\left(J_\nu^\alpha f\right)(x) = \left(1 + |x|^2\right)^{-\alpha/2} F_\nu(f)(x), \quad x \in \mathbb{R}_+^n. \tag{4}$$

These potentials are known as important technical tools in Fourier and Fourier–Bessel harmonic analysis (More information about these potentials can be found in [1–3,5,10–12]).

In this paper we investigate the approximation properties of the family $A^\alpha f$, when the parameter $\alpha > 0$ tends to zero. The paper is organized as follows. Section 2 contains basic notations, definitions and auxiliary lemmas. In particular, the notion of the "admissible bunch" of operators is introduced and some examples are given in the section. The main results of the paper are given in Section 3. This section is devoted to the investigation of approximation properties of the family $(A^\alpha f)(x)$ as $\alpha \to 0^+$. The order of approximation of the Lipschitz functions is also studied. Moreover, some applications to the Riesz and the Bessel potentials generated by the Euclidean and generalized translations are given. It should also be mentioned that the approximation properties of the classical Riesz and Bessel potentials have been studied by Kurokawa [7] before.

2. Preliminaries and auxiliary lemmas

Let $L_p \equiv L_p\left(\mathbb{R}^n, dm\right)$ be the space of m-measurable functions such that

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p\, dm(x)\right)^{1/p} < \infty, \quad 1 \leqslant p < \infty,$$

and let $C_0 \equiv C_0(\mathbb{R}^n)$ be the class of all continuous functions on $\mathbb{R}^n$ vanishing at infinity. We will assume that the set of all compactly supported continuous functions is dense in $L_p(\mathbb{R}^n, dm)$ (e.g., this is the case when m is a Borel measure).