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Energy and discrepancy of rotationally invariant determinantal point processes in high dimensional spheres[☆]

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ABSTRACT

We study expected Riesz s-energies and linear statistics of some determinantal processes on the sphere $\mathbb{S}^d$. In particular, we compute the expected Riesz and logarithmic energies of the determinantal processes given by the reproducing kernel of the space of spherical harmonics. This kernel defines the so called harmonic ensemble on $\mathbb{S}^d$. With these computations we improve previous estimates for the discrete minimal energy of configurations of points in the sphere. We prove a comparison result for Riesz 2-energies of points defined through determinantal point processes associated with isotropic kernels. As a corollary we get that the Riesz 2-energy of the harmonic ensemble is optimal among ensembles defined by isotropic kernels with the same trace. Finally, we study the variance of smooth and rough linear statistics for the harmonic ensemble and compare the results with the variance for the spherical ensemble (in $\mathbb{S}^2$).

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1. Introduction

Let $\mathbb{S}^d$ be the unit sphere in the Euclidean space $\mathbb{R}^{d+1}$ and let μ be the normalized Lebesgue surface measure. We study Riesz s-energies and the uniformity (discrepancy and separation) of random configurations of points on the sphere $\mathbb{S}^d$ given by some determinantal point processes.

1.1. Riesz energies

For a given collection of points $x_1, \ldots, x_n \in \mathbb{S}^d$ and $s > 0$ the discrete s-energy associated with the tuple $x = (x_1, \ldots, x_n)$ is

$$E_s(x) = \sum_{i \neq j} \frac{1}{\|x_i - x_j\|^s}.$$

The minimal Riesz s-energy is the value $\mathcal{E}(s, n) = \inf_x E_s(x)$, where x runs through the n-point subsets of $\mathbb{S}^d$. The limiting case $s = 0$, given (through $(t^s - 1)/s \to \log t$ when $s \to 0$) by

$$E_0(x) = \sum_{i \neq j} \log \frac{1}{\|x_i - x_j\|},$$

is the discrete logarithmic energy associated with x and $\mathcal{E}(0, n) = \inf_x E_0(x)$, is the minimal discrete logarithmic energy for n points on the sphere.

The asymptotic behavior of these energies, in the spherical and in other settings, has been extensively studied. See for example the survey papers [10,9]. One problem in studying these quantities is to get computable examples. It is natural then to study random configurations of points and try to estimate asymptotically their energies on average. The next natural question is how to get good random configurations on the sphere. It is clear that uniform random points are not good candidates to have low energies because there is no local repulsion between points and the sets exhibit clumping.

A method to get better distributed random points is to take sets of zeros of certain random holomorphic polynomials on the plane with independent coefficients and transport them to the sphere via the stereographic projection. As the zeros repel each other, the configurations exhibit no clumping. This idea was used in [4] and the authors managed to get, in $\mathbb{S}^2$, the average behavior of the logarithmic energy (other relations between the logarithmic energy and polynomial roots are known, see [35]). See the works of Zelditch et al. [34,41,42,15] for an extension to several complex variables.

We consider instead random sets of points given by a determinantal process. Random points drawn from a determinantal process exhibit local repulsion, they can be built in any dimension and they are computationally feasible, as proven in [6] and implemented in [31].

1.2. Determinantal processes

In this section we follow [6, Chap. 4]. See also [3,31] or [37].

We denote as $\mathcal{X}$ a simple random point process in $\mathbb{S}^d$, that is a random discrete subset of $\mathbb{S}^d$ (the definition of a point process which is not "simple" is a little more involved, see [6, Sec. 1.2]). A way to describe the process is to specify the random variable counting the number of points of the process in D, for all Borel sets $D \subset \mathbb{S}^d$. We denote this random variable as $\mathcal{X}(D)$ or n_D. In many cases the point process is conveniently characterized by the so-called joint intensity functions, see [21,22].

The joint intensities $\rho_k(x_1, \ldots, x_k)$ are functions defined in $(\mathbb{S}^d)^k$ such that for any family of mutually disjoint subsets $D_1, \ldots, D_k \subset \mathbb{S}^d$

$$\mathbb{E}\left[\mathcal{X}(D_1) \cdots \mathcal{X}(D_k)\right] = \int_{D_1 \times \cdots \times D_k} \rho_k(x_1, \ldots, x_k) d\mu(x_1) \ldots d\mu(x_k),$$

we assume that $\rho_k(x_1, \ldots, x_k) = 0$ when $x_i = x_j$ for $i \neq j$.