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Digital nets with infinite digit expansions and construction of folded digital nets for quasi-Monte Carlo integration[☆]

Takashi Goda[a,*], Kosuke Suzuki[b,1], Takehito Yoshiki[b]

[a] *Graduate School of Engineering, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan*

[b] *Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan*

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A B S T R A C T

In this paper we study quasi-Monte Carlo integration of smooth functions using digital nets. We fold digital nets over $\mathbb{Z}_b$ by means of the b-adic tent transformation, which has recently been introduced by the authors, and employ such *folded digital nets* as quadrature points. We first analyze the worst-case error of quasi-Monte Carlo rules using folded digital nets in reproducing kernel Hilbert spaces. Here we need to permit digital nets with "infinite digit expansions", which are beyond the scope of the classical definition of digital nets. We overcome this issue by considering the infinite product of cyclic groups and the characters on it. We then give an explicit means of constructing good folded digital nets as follows: we use higher order polynomial lattice point sets for digital nets and show that the component-by-component construction can find good *folded higher order polynomial lattice rules* that achieve the optimal convergence rate of the worst-case error in certain Sobolev spaces of smoothness of arbitrarily high order.

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* Corresponding author.

E-mail addresses: goda@frcer.t.u-tokyo.ac.jp (T. Goda), ksuzuki@ms.u-tokyo.ac.jp (K. Suzuki), yosiki@ms.u-tokyo.ac.jp (T. Yoshiki).

1 JSPS research fellow.

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1. Introduction

Quasi-Monte Carlo (QMC) integration of a real-valued function f defined over the s-dimensional unit cube is given by

$$Q(f;P) := \frac{1}{|P|}\sum_{\boldsymbol{x}\in P} f(\boldsymbol{x}),$$

where $P \subset [0,1]^s$ is a point set and $|P|$ denotes the cardinality of P, to approximate the integral

$$I(f) := \int_{[0,1]^s} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

as accurately as possible. As one of the main families of QMC point sets, digital nets and sequences have been extensively studied in the literature, see for instance [8,15]. We shall discuss the definition of digital nets in Section 2.3. There have been many good explicit constructions of digital nets and sequences, including those proposed by Sobol', Faure, Niederreiter, Niederreiter and Xing as well as others, see [8, Section 8] for more information. These point sets generally hold good properties of uniform distribution modulo one. The typical convergence rate of the QMC integration error $|I(f) - Q(f;P)|$ using these point sets is $O(|P|^{-1+\varepsilon})$ with arbitrarily small $\varepsilon > 0$.

Our goal of this paper is to give an explicit means of constructing good deterministic point sets for QMC integration of smooth functions in certain Sobolev spaces $\mathcal{H}_\alpha$ of smoothness of arbitrarily high order $\alpha \geq 2$. For such smooth functions, it is possible to achieve higher order convergence of $O(|P|^{-\alpha+\varepsilon})$ (with arbitrarily small $\varepsilon > 0$) of the QMC integration error by using *higher order* digital nets [1,4]. The explicit construction of higher order digital nets introduced in [4] uses digital nets whose number of components is a multiple of the dimension, and interlaces them digitally in a certain way. Another construction of higher order digital nets, known under the name of *higher order polynomial lattice point sets* (HOPLPSs), is introduced in [7] by generalizing the definition of polynomial lattice point sets, which was originally given in [16]. Recently, one of the authors has utilized original polynomial lattice point sets as interlaced components in the former construction principle, and has proved that it is possible to obtain good *interlaced polynomial lattice point sets* (IPLPSs) for higher order digital nets [10], see also [12].

The most important advantage of IPLPSs over HOPLPSs lies in the construction cost. Fast component-by-component (CBC) construction requires $O(s\alpha|P|\log|P|)$ arithmetic operations using $O(|P|)$ memory for IPLPSs [10], whereas requiring $O(s\alpha|P|^\alpha \log|P|)$ arithmetic operations using $O(|P|^\alpha)$ memory for HOPLPSs [2]. In order to reduce the construction cost for HOPLPSs, one of the authors considered applying a random digital shift and then folding the resulting point sets by using the tent transformation in [11]. (We note that the tent transformation was originally used for lattice rules in [14].) The obtained cost for the fast CBC construction becomes $O(s\alpha|P|^{\alpha/2}\log|P|)$ arithmetic operations using $O(|P|^{\alpha/2})$ memory. This is a generalization of the study in [3]. However, this result not only restricts the base b to 2, but also needs a randomization by a random digital shift. Thus, we cannot construct good *deterministic* point sets in this way in contrast to [2,10].

Regarding the restriction of the base, the authors have recently introduced the *b-adic tent transformation* (b-TT) in [13] for any positive integer $b \geq 2$, by generalizing the original (dyadic) tent transformation, and studied the mean square worst-case error of *digitally shifted and then folded* digital nets in reproducing kernel Hilbert spaces. As a continuation of the study in [13], we resolve the above concerns on [11] in this paper.

We first consider QMC point sets which are obtained by folding digital nets by means of the b-TT, and study the worst-case error of QMC rules using such *folded digital nets* in reproducing kernel Hilbert spaces. In our analysis, we need to permit digital nets with "infinite digit expansions", which are beyond the scope of the classical definition of digital nets, see for example [8, Chapter 4] and [15, Chapter 4]. To overcome this issue, we consider infinite products of cyclic groups G and the characters on G, and define digital nets in G^s by using infinite-column generating matrices, i.e., generating matrices whose each column can contain infinitely many entries different from zero,