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# Hyperbolic cross approximation in infinite dimensions

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ABSTRACT

We give tight upper and lower bounds of the cardinality of the index sets of certain hyperbolic crosses which reflect mixed Sobolev–Korobov-type smoothness and mixed Sobolev-analytic-type smoothness in the infinite-dimensional case where specific summability properties of the smoothness indices are fulfilled. These estimates are then applied to the linear approximation of functions from the associated spaces in terms of the $\varepsilon$-dimension of their unit balls. Here, the approximation is based on linear information. Such function spaces appear for example for the solution of parametric and stochastic PDEs. The obtained upper and lower bounds of the approximation error as well as of the associated $\varepsilon$-complexities are completely independent of any parametric or stochastic dimension. Moreover, the rates are independent of the parameters which define the smoothness properties of the infinite-variate parametric or stochastic part of the solution. These parameters are only contained in the order constants. This way, linear approximation theory becomes possible in the infinite-dimensional case and corresponding infinite-dimensional problems get tractable.



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## 1. Introduction

The efficient approximation of a function of infinitely many variables is an important issue in a lot of applications in physics, finance, engineering and statistics. It arises in uncertainty quantification, computational finance and computational physics and is encountered for example in the numerical treatment of path integrals, stochastic processes, random fields and stochastic or parametric PDEs. While the problem of quadrature of functions in weighted Hilbert spaces with infinitely many variables has recently found a lot of interest in the information based complexity community, see e.g. [3,15–17,29,30,34,35,42,44,50–52,60], there is much less literature on approximation. So far, the approximation of functions in weighted Hilbert spaces with infinitely many variables has been studied for a properly weighted $L_2$-error norm[1] in [59]. In any case, a reproducing kernel Hilbert space $H_K$ with kernel $K = \sum_u \gamma_u k_u$ is employed with $|u|$-dimensional kernels $k_u$ where $u$ varies over all finite subsets of $\mathbb{N}$. It involves a sequence of weights $\gamma = (\gamma_u)$ that moderate the influence of terms which depend on the variables associated with the finite-dimensional index sets $u \subset \{1, 2, \ldots\} = \mathbb{N}$. Weighted spaces had first been suggested for the finite-dimensional case in [52], see also [53,54]. For further details, see [18] and the references cited therein. The approximation of functions with anisotropically weighted Gaussian kernels has been studied in [27].

Moreover, there is work on sparse grid integration and approximation, see [8] for a survey and bibliography. It recently has found applications in uncertainty quantification for stochastic and parametric PDEs, especially for non-intrusive methods, compare [4,5,11–14,33,36,37,43,45,46]. There, for the stochastic or parametric part of the problem, an anisotropic sparse grid approximation or quadrature is constructed either a priori from knowledge of the covariance decay of the stochastic data or a posteriori by means of dimension-adaptive refinement. This way, the infinite-dimensional case gets truncated dynamically to finite dimensions while the higher dimensions are trivially resolved by the constant one. Here, in contrast to the above-mentioned approach using a weighted reproducing kernel Hilbert space, one usually relies on spaces with smoothness of increasing order, either for the mixed Sobolev smoothness situation or for the analytic setting. Thus, as already noticed in [49], one may have two options for obtaining tractability: either by using decaying weights or by using increasing smoothness.

Sparse grids and hyperbolic crosses promise to break the curse of dimensionality which appears for conventional approximation methods, at least to some extent. However, the approximation rates and cost complexities of conventional sparse grids for isotropic mixed Sobolev regularity still involve logarithmic terms which grow exponentially with the dimension. In [23,31] it could be shown that the rate of the approximation error and the cost complexity get completely independent of the dimension for the case of anisotropic mixed Sobolev regularity with sufficiently fast rising smoothness indices. This also follows from results on approximation for anisotropic mixed smoothness, see, e.g., [22,55] for details. But the constants in the bounds for the approximation error and the cost rate could not be estimated sharply and still depend on the dimension $d$. Therefore, this result cannot straightforwardly be extended to the infinite-dimensional case, i.e. to the limit of $d$ going to $\infty$.

This will be achieved in the present paper. To this end, we rely on the infinite-variate space $\mathcal{H}$ which is the tensor product $\mathcal{H} := H^\alpha(\mathbb{G}^m) \otimes K^{\mathbf{r}}(\mathbb{D}^\infty)$ of the Sobolev space $H^\alpha(\mathbb{G}^m)$ and the infinite-variate space $K^{\mathbf{r}}(\mathbb{D}^\infty)$ of mixed smoothness with varying Korobov-type smoothness indices $\mathbf{r} = r_1, r_2, \ldots$, or we rely on the tensor product $\mathcal{H} := H^\alpha(\mathbb{G}^m) \otimes A^{\mathbf{r},p,q}(\mathbb{D}^\infty)$ of $H^\alpha(\mathbb{G}^m)$ with the infinite-variate space $A^{\mathbf{r},p,q}(\mathbb{D}^\infty)$ of mixed smoothness with varying analytic-type smoothness indices $\mathbf{r} = r_1, r_2, \ldots$ (and $p$ and $q$ entering algebraic prefactors). The approximation error is measured in the tensor product Hilbert space $\mathcal{G} := H^\beta(\mathbb{G}^m) \otimes L_2(\mathbb{D}^\infty, \mu)$ with $\beta \geq 0$, which is isomorphic to the Bochner space $L_2(\mathbb{D}^\infty, H^\beta(\mathbb{G}^m))$. Here, $\mathbb{G}$ denotes either the unit circle (one-dimensional torus) $\mathbb{T}$ in the periodic case or the interval $\mathbb{I} := [-1, 1]$ in the nonperiodic case. Furthermore, $\mathbb{D}$ is either $\mathbb{T}$, $\mathbb{I}$ or $\mathbb{R}$, depending on the respective situation under consideration. Altogether, $\mathbb{G}^m$ denotes the $m$-fold (tensor-product) domain where the $m$-dimensional physical coordinates live, whereas $\mathbb{D}^\infty$

[1] There is also [58,61,62], where however a norm in a special Hilbert space was employed such that the approximation problem indeed got easier than the integration problem.