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Enlarging the domain of starting points for Newton's method under center conditions on the first Fréchet-derivative

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ABSTRACT

We analyze the semilocal convergence of Newton's method under center conditions on the first Fréchet-derivative of the operator involved. We see that we can extend the known results so far, since we provide different starting points from the point where the first Fréchet-derivative is centered (that is the situation usually considered by other authors), so that the domain of starting points is enlarged for Newton's method. We also illustrate the theoretical results obtained with some mildly nonlinear elliptic equations.

1. Introduction

A large number of problems in applied mathematics and engineering are solved by finding the solutions of nonlinear equations. The solutions of these equations can rarely be found in a closed form, so that we usually look for numerical approximations of these solutions. In the majority of situations, the most commonly used solution methods are iterative that provide a sequence of approximations that converges to a solution of an equation. The most popular iterative method for approximating these solutions is undoubtedly Newton's method.

To give sufficient generality to the problem of approximating a solution of a nonlinear equation by Newton's method, we consider equations of the form $F(x) = 0$, where F is a nonlinear operator, $F : \Omega \subseteq X \longrightarrow Y$, defined on a nonempty open convex domain Ω of a Banach space X with values

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in a Banach space Y, which is also usually known as the Newton–Kantorovich method and whose algorithm is

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_n = x_{n-1} - [F'(x_{n-1})]^{-1}F(x_{n-1}), \quad n \in \mathbb{N}. \end{cases} \tag{1}$$

In this paper, we focus our attention on the analysis of the semilocal convergence of Newton's method, that is based on demanding conditions to the initial approximation x_0, from certain conditions on the operator F, and provide the so-called domain of parameters corresponding to the conditions required to the initial approximation that guarantee the convergence of sequence (1) to a solution x^* of $F(x) = 0$.

If we focus on the type of conditions demanded to the operator F, under which the convergence of Newton's method is guaranteed, we see that conditions on F' are imposed [5,9,15,20], as well as conditions on F'' [10,12,18] and even conditions on successive derivatives of F [1,2,7]. In this work, conditions on F' are imposed. The most popular semilocal convergence result of this type for Newton's method is the variant of the Newton–Kantorovich theorem [18] given by Ortega in [20], which is established under the following conditions:

(A1) There exists $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$, for some $x_0 \in \Omega$, with $\|\Gamma_0\| \le \beta$ and $\|\Gamma_0 F(x_0)\| \le \eta$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y to X.
(A2) There exists a constant $K \ge 0$ such that $\|F'(x) - F'(y)\| \le K\|x - y\|$ for $x, y \in \Omega$.
(A3) $K\beta\eta \le \frac{1}{2}$.

Ortega's result guarantees the semilocal convergence of Newton's method if condition (A3) is satisfied, once the parameters β, η and K of (A1) and (A2) are known, and $B(x_0, t^*) \subset \Omega$, where $t^* = \frac{1-\sqrt{1-2K\beta\eta}}{K\beta}$ is the smallest positive zero of the polynomial $p(t) = \frac{K}{2}t^2 - \frac{t}{\beta} + \frac{\eta}{\beta}$. Observe that condition (A1), required to the initial approximation x_0, define the parameters β and η, and condition (A2), required to the operator F, define the fixed parameter K.

Obviously, given $x_0 \in \Omega$, to guarantee the semilocal convergence of Newton's method from Ortega's result, the pair $(K\beta, \eta)$, associated with x_0, must belong to the domain

$$D = \left\{ (x, y) \in \mathbb{R}^2 : xy \le \frac{1}{2} \right\},$$

which we call domain of parameters associated with Ortega's result. That is: given a point $x_0 \in \Omega$, we obtain the parameters β and η, check if $(K\beta, \eta) \in D$ and if $(K\beta, \eta) \in D$, then the convergence of Newton's method to a solution of the equation $F(x) = 0$ is guaranteed from starting at x_0.

On the other hand, we can modify condition (A2) of Ortega by centering it in the following way:

(B2) There exist $\widetilde{x} \in \Omega$ and $\widetilde{K} \in \mathbb{R}_+$ such that $\|F'(x) - F'(\widetilde{x})\| \le \widetilde{K}\|x - \widetilde{x}\|, \ x \in \Omega$.

A consequence of this modification of (A2) leads us to modify condition (A3). In addition, as condition (B2) is milder than condition (A2), then the corresponding variant of condition (A3) is more restrictive. In the known results so far, in which condition (B2) is imposed to the operator F' instead of (A2), $x_0 = \widetilde{x}$ is considered, as we can see in [13], where the condition

(A3b) $\widetilde{K}\beta\eta \le \frac{14-4\sqrt{6}}{25} = 0.1680816\ldots$

is required, instead of (A3), to guarantee the semilocal convergence of Newton's method. In this case, the following semilocal convergence result for Newton's method is given in [13].

Theorem 1. *Let $F : \Omega \subseteq X \longrightarrow Y$ be a once continuously differentiable operator defined on a nonempty open convex domain Ω of a Banach space X with values in a Banach space Y. Suppose that conditions (A1)–(B2)–(A3b) with $\widetilde{x} = x_0$ are satisfied. If $B(x_0, \rho_1) \subset \Omega$, where $\rho_1 = \frac{2+5h-\sqrt{(2+5h)^2-48h}}{12\widetilde{K}\beta}$ and $h = \widetilde{K}\beta\eta$, then Newton's sequence, given by (1), converges to a solution x^* of the equation $F(x) = 0$, starting at x_0, and $x_n, x^* \in \overline{B(x_0, \rho_1)}$, for all $n = 0, 1, 2, \ldots$ Moreover, the solution x^* is unique in $B(x_0, \rho_2) \cap \Omega$, where $\rho_2 = \frac{22-5h+\sqrt{(2+5h)^2-48h}}{12\widetilde{K}\beta}$.*