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General convergence theorems for iterative processes and applications to the Weierstrass root-finding method

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ABSTRACT

In this paper, we prove some general convergence theorems for the Picard iteration in cone metric spaces over a solid vector space. As an application, we provide a detailed convergence analysis of the Weierstrass iterative method for computing all zeros of a polynomial simultaneously. These results improve and generalize existing ones in the literature.

1. Introduction

In the first part of the paper, we study the convergence of the iterative processes of the type

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $T : D \subset X \to X$ is an iteration function in a cone metric space (X, d) over a solid vector space $(Y, \preceq)$. Cone metric spaces have a long history (see Collatz [3], Zabrejko [43], Janković, Kadelburg and Radenović [10], Proinov [29] and references therein). For an overview of the theory of cone metric spaces over a solid vector space, we refer the reader to [29] and [31, Section 2].

In the second part of the paper, we study the convergence of the famous Weierstrass method [39] for computing all zeros of a polynomial simultaneously. This method was introduced and studied for the first time by Weierstrass in 1891. In 1960–1966, the method was rediscovered by Durand [6]

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(in implicit form), Dochev [4], Kerner [11] and Prešić [22]. For this reason, it is also known as 'Durand–Kerner method', 'Weierstrass–Dochev method', etc. For an overview of iterative methods for simultaneous finding of polynomial zeros, we refer the reader to [34,12,14,18].

Throughout this paper, $(\mathbb{K}, |\cdot|)$ denotes an arbitrary normed (valued) field with absolute value $|\cdot|$, and $\mathbb{K}[z]$ denotes the ring of polynomials in one variable z over $\mathbb{K}$. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. We consider the zeros of f as a vector in $\mathbb{K}^n$. More precisely, a vector $\xi \in \mathbb{K}^n$ is said to be a *root-vector* of f if

$$f(z) = a_0 \prod_{i=1}^{n} (z - \xi_i) \quad \text{for all } z \in \mathbb{K}, \tag{1.2}$$

where $a_0 \in \mathbb{K}$. Obviously, f has a root-vector in $\mathbb{K}^n$ if and only if f splits in $\mathbb{K}$. Recall that the Weierstrass method is defined by the following iteration

$$x^{k+1} = x^k - W(x^k), \quad k = 0, 1, 2, \ldots, \tag{1.3}$$

where $W : \mathcal{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ is defined by $W(x) = (W_1(x), \ldots, W_n(x))$ with

$$W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \neq i} (x_i - x_j)} \quad (i = 1, \ldots, n), \tag{1.4}$$

where a_0 is the leading coefficient of f and $\mathcal{D}$ is the set of all vectors in $\mathbb{K}^n$ with distinct components. The operator W is called the *Weierstrass correction*. Sometimes we write W_f instead of W to indicate that the operator W is generated by f. It is easy to see that the Weierstrass correction W_f is invariant with respect to multiplication of f by a non-zero constant $c \in \mathbb{K}$. Obviously, the Weierstrass iteration (1.3) can be represented in the form (1.1) with the iteration function $T : \mathcal{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ defined by

$$T(x) = x - W(x). \tag{1.5}$$

The aim of this paper is twofold. First, we present some general convergence theorems with error estimates for the Picard iteration (1.1). These results extend some of the results in [27,28]. Second, using these results we provide a detailed convergence analysis of the Weierstrass method (1.3). The new results for the Weierstrass method improve the corresponding results of [2,4,23,45,13,46,37,44,38,19,17,21,36,41,7,20,24,32].

The paper is structured as follows:

In Section 2, we present some preliminary results and notations that will be useful in the sequel.

In Section 3, we establish two general convergence theorems with error estimates for iterated contractions at a point in cone metric spaces. The first one extends Theorem 3.6 of [27].

In Section 4, we establish two general semilocal convergence theorems with error estimates for iterative processes of the type (1.1). These results extend Theorems 5.4 and 5.6 of [28]. As a consequence we obtain a convergence theorem with error estimates for iterated contractions in cone metric spaces, which extends Theorem 6.5 of [28]. All results in this section are generalizations of the Banach Contraction Principle [1] as well as of the Iterated Contraction Principle given in [15, Chap. 12] and [29].

In Section 5, we present some inequalities in $\mathbb{K}^n$ and notations which will be useful in the next sections.

In Section 6, we obtain a local convergence theorem with error estimates for the Weierstrass method which improves the results of Dochev [4], Kyurkchiev and Markov [13], Yakoubsohn [41] and Proinov and Petkova [32].

In Section 7, we obtain another local convergence theorem with error estimates for the Weierstrass method which improves and generalizes the results of Wang and Zhao [37], Tilli [36] and Han [7].

In Section 8, we prove a new convergence theorem for the Weierstrass method under computationally verifiable initial conditions. The main result of this section generalizes, improves and complements all previous results in this area, which are due to Prešić [23], Zheng [45,46], Wang and Zhao [44,38], Petković, Carstensen and Trajković [19], Petković [17], Petković, Herceg and Ilić [21], Batra [2],