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L_p- and $S^r_{p,q}B$-discrepancy of the symmetrized van der Corput sequence and modified Hammersley point sets in arbitrary bases

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ABSTRACT

We study the local discrepancy of a symmetrized version of the well-known van der Corput sequence and of modified two-dimensional Hammersley point sets in arbitrary base b. We give upper bounds on the norm of the local discrepancy in Besov spaces of dominating mixed smoothness $S^r_{p,q}B([0,1)^s)$, which will also give us bounds on the L_p-discrepancy. Our sequence and point sets will achieve the known optimal order for the L_p- and $S^r_{p,q}B$-discrepancy. The results in this paper generalize several previous results on L_p- and $S^r_{p,q}B$-discrepancy estimates and provide a sharp upper bound on the $S^r_{p,q}B$-discrepancy of one-dimensional sequences for $r > 0$. We will use the b-adic Haar function system in the proofs.

1. Introduction and statement of the results

For an N-element point set $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}\}$ in the s-dimensional unit interval $[0,1)^s$ the local discrepancy $D_N(\mathcal{P}, \boldsymbol{t})$ is defined as

$$D_N(\mathcal{P}, \boldsymbol{t}) := \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{[\mathbf{0},\boldsymbol{t})}(\boldsymbol{x}_n) - \prod_{i=1}^{s} t_i.$$

In this expression, for $\boldsymbol{t} = (t_1, \dots, t_s) \in [0,1]^s$, the notation $[\mathbf{0}, \boldsymbol{t})$ means the s-dimensional interval $[0, t_1) \times \cdots \times [0, t_s)$ with volume $\prod_{i=1}^{s} t_i$ and $\mathbf{1}_I$ denotes the indicator function of the interval

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$I \subseteq [0,1]^s$. For an infinite sequence $\mathcal{S} = (\boldsymbol{x}_n)_{n\geq 0}$ of elements in $[0,1)^s$ the local discrepancy $D_N(\mathcal{S}, \boldsymbol{t})$ is defined as the local discrepancy of its first N elements.

We denote the norm of the local discrepancy in a normed space X of functions on $[0,1)^s$ by $\|D_N(\mathcal{P}, \cdot) \mid X\|$, where we must require $D_N(\mathcal{P}, \cdot) \in X$.

In this paper we are interested in particular normed spaces, namely the $L_p([0,1)^s)$ spaces and the Besov spaces $S^r_{p,q}B([0,1)^s)$ of dominating mixed smoothness. The definition of the latter is given in Section 4. For $p \in [1,\infty]$, the $L_p([0,1)^s)$ space is defined as the collection of all functions f on $[0,1)^s$ with finite $L_p([0,1)^s)$ norm, which for $1 \leq p < \infty$ is defined as

$$\left\| f \mid L_p([0,1)^s) \right\| := \left(\int_{[0,1)^s} |f(\boldsymbol{t})|^p \mathrm{d}\,\boldsymbol{t} \right)^{\frac{1}{p}},$$

and for $p = \infty$ is given by

$$\left\| f \mid L_\infty([0,1)^s) \right\| := \sup_{\boldsymbol{t}\in[0,1]^s} |f(\boldsymbol{t})|.$$

We speak of $\left\|D_N(\mathcal{P}, \cdot) \mid L_p([0,1)^s)\right\|$ and $\left\|D_N(\mathcal{P}, \cdot) \mid S^r_{p,q}B([0,1)^s)\right\|$ as the L_p- and the $S^r_{p,q}B$-discrepancy of a point set $\mathcal{P} \in [0,1)^s$, respectively. An analogous notation is used for sequences $\mathcal{S} \in [0,1)^s$. The L_∞-discrepancy is the well-studied star discrepancy, but in this paper we will assume that $p \in [1,\infty)$.

The L_p-discrepancy is a quantitative measure for the irregularity of distribution of a sequence modulo one, see e.g. [10,19,25]. It is also related to the worst-case integration error of a quasi-Monte Carlo rule, see e.g. [7,21,26]. The $S^r_{p,q}B$-discrepancy is related to the errors of quasi-Monte Carlo algorithms for numerical integration on spaces of dominating mixed smoothness, see e.g. [31].

It is well known that for every $p \in (1,\infty)$ and for all $s \in \mathbb{N}$ there exist positive numbers $c_{p,s}$ and $c'_{p,s}$ with the property that for every $N \geq 2$ any N-element point set $\mathcal{P}$ in $[0,1)^s$ satisfies

$$\left\|D_N(\mathcal{P}, \cdot) \mid L_p([0,1)^s)\right\| \geq c_{p,s} \frac{(\log N)^{\frac{s-1}{2}}}{N}, \tag{1}$$

and for every sequence $\mathcal{S}$ in $[0,1)^s$ we have

$$\left\|D_N(\mathcal{S}, \cdot) \mid L_p([0,1)^s)\right\| \geq c'_{p,s} \frac{(\log N)^{\frac{s}{2}}}{N} \quad \text{for infinitely many } N \in \mathbb{N}, \tag{2}$$

where log denotes the natural logarithm. The inequality (1) was shown by Roth [28] for $p = 2$ (and therefore for $p \in (2,\infty)$ because of the monotonicity of the L_p norms) and Schmidt [29] for $p \in (1,2)$. Proinov [27] could prove (2) based on the results of Roth and Schmidt. Halász [14] showed that the bounds (1) and (2) also hold for the L_1-discrepancy of two-dimensional point sets and one-dimensional sequences, respectively. There exist point sets in every dimension s with the order of the L_p-discrepancy of $(\log N)^{\frac{s-1}{2}}/N$ for $p \in (1,\infty)$ (see [2] for the first existence result), which shows that the lower bound given in (1) is sharp. Chen and Skriganov [3] gave for the first time for every integer $N \geq 2$ and every dimension $s \in \mathbb{N}$, explicit constructions of finite N-element point sets in $[0,1)^s$ whose L_2-discrepancy achieves an order of convergence of $(\log N)^{\frac{s-1}{2}}/N$. The result in [3] was extended to the L_p-discrepancy for $p \in (1,\infty)$ by Skriganov [30]. The inequality (2) is also sharp for one-dimensional sequences (see e.g. [18]). Moreover, it is sharp for the L_2-discrepancy in all dimensions (see [9,8]). Showing sharpness for all $p \in (1,\infty)$ in all dimensions is currently work in progress.

There are also known lower and upper bounds for the $S^r_{p,q}B$-discrepancy in arbitrary dimensions. Triebel, who initiated the study of the local discrepancy in other spaces such as the Besov spaces and Triebel–Lizorkin spaces of dominating mixed smoothness in [31,32], showed that for all $1 \leq p, q \leq \infty$ and $r \in \mathbb{R}$ satisfying $\frac{1}{p} - 1 < r < \frac{1}{p}$ and $q < \infty$ if $p = 1$ and $q > 1$ if $p = \infty$ there exists a constant $c_1 > 0$ such that for any $N \geq 2$ the local discrepancy of any N-element point set $\mathcal{P}$ in $[0,1)^s$ satisfies

$$\left\|D_N(\mathcal{P}, \cdot) \mid S^r_{p,q}B([0,1)^s)\right\| \geq c_1 N^{r-1} (\log N)^{\frac{s-1}{q}}. \tag{3}$$