Journal of Complexity 32 (2016) 122–136

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

Approximation of piecewise Hölder functions from inexact information

Paweł M. Morkisz [a,*], Leszek Plaskota [b]

[a] AGH University of Science and Technology, Faculty of Applied Mathematics, Al. A. Mickiewicza 30, 30-059 Krakow, Poland

[b] Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, ul. S. Banacha 2, 02-097 Warsaw, Poland

ARTICLE INFO

Article history:
Received 5 March 2015
Accepted 9 September 2015
Available online 25 September 2015

Keywords:
Numerical approximation
Noisy data
Piecewise smooth functions
Adaptive algorithms

ABSTRACT

We study the L^p-approximation of functions f consisting of two smooth pieces separated by an (unknown) singular point s_f; each piece is r times differentiable and the rth derivative is Hölder continuous with exponent ϱ. The approximations use n inexact function values $y_i = f(x_i) + e_i$ with $|e_i| \le \delta$. Let $1 \le p < \infty$. We show that then the minimal worst case error is proportional to $\max(\delta, n^{-(r+\varrho)})$ in the class of functions with uniformly bounded both the Hölder coefficients and the discontinuity jumps $|f(s_f^+) - f(s_f^-)|$. This error is achieved by an algorithm that uses a new adaptive mechanism to approximate s_f, where the number of adaptively chosen points x_i is only $\mathcal{O}(\ln n)$. The use of adaption, $p < \infty$, and the uniform bound on the Hölder coefficients and the discontinuity jumps are crucial. If we restrict the class even further to continuous functions, then the same worst case result can be achieved also for $p = \infty$ using no more than $(r-1)_+$ adaptive points. The results generalize earlier results that were obtained for exact information (where $\delta = 0$) and $f^{(r)}$ piecewise Lipschitz (where $\varrho = 1$).

1. Introduction

The problem of approximating smooth functions based on information about n function values has been very well studied. Much less explored, but more important for applications, is approximation of

* Corresponding author.
E-mail addresses: morkiszp@agh.edu.pl (P. Morkisz), leszekp@mimuw.edu.pl (L. Plaskota).

http://dx.doi.org/10.1016/j.jco.2015.09.002
0885-064X/

functions that are piecewise smooth only. Note that a function f is piecewise smooth if its domain can be divided into subdomains such that f is smooth in each subdomain.

We say that a scalar function g is (r, ϱ)-smooth in an interval $[a, b]$ if $g \in C^r([a, b])$ and $g^{(r)}$ is Hölder continuous with exponent $\varrho \in (0, 1]$. We consider the space $F_{r,\varrho}$ of T-periodic functions f that consist of two (r, ϱ)-smooth pieces separated by an unknown singular point s_f. We stress that the periodicity is assumed only to simplify the presentation; after some technical modifications, all the results hold also for nonperiodic functions, cf. [15].

The L^p-approximation in the space $F_{r,\varrho}$ (periodic or non-periodic) based on function evaluations has been so far studied assuming Lipschitz continuity of the rth derivatives (that is, for $\varrho = 1$) and exact information about function values, see, e.g., [1,10,15–17]. In [16], classes of piecewise smooth functions contained in $F_{r,1}$ were identified for which the minimal worst case error is proportional to $n^{-(r+1)}$, which is also the minimal worst case error for the (globally) smooth functions, cf. [11,21]. This holds for $p < \infty$ and for the Skorohod metric, but not for $p = \infty$. However, to obtain such error level, the use of adaption is necessary. Optimal algorithms use an adaptive mechanism to detect essential singularities s_f, i.e., those with 'large' discontinuity jumps of the derivatives of f. This mechanism basically relies on using divided differences on uniform mesh, and the fact that the largest divided difference covers s_f if the mesh-size is small enough. Approximation of continuous functions from $F_{r,\varrho} \cap C$ was studied in [15] where it was shown that then the worst case error of level $n^{-(r+1)}$ can be obtained even for the uniform approximation. A related work has also been done for other problems such as numerical integration [14], solving ODEs [6–9], or solving SDEs [18,19]. For approximation of piecewise smooth functions using other methods (including spectral methods) see, e.g., [2–5,20].

In the present paper we generalize the results of [15,16] in two directions. We assume that the Hölder exponent $\varrho \in (0, 1]$ and, what is more important, information is obtained with some error. This means that instead of the exact values $f(x_i)$ for $i = 1, 2, \ldots, n$ one obtains perturbed values $y_i = f(x_i) + e_i$ where $|e_i| \le \delta$, cf. [12,13]. It is not difficult to see that in the presence of perturbations the minimal worst case error is at least proportional to $\min(\delta, n^{-(r+\varrho)})$. The question is whether this worst case error can be achieved by a particular algorithm for reasonable classes $\mathcal{F} \subset F_{r,\varrho}$. The answer is not obvious since the adaptive algorithms developed in [15,16] do not work well for inexact information. Indeed, in the presence of noise the divided differences calculated from noisy function values do not say anything about the location of s_f if the mesh-size is too small. On the other hand, a localization of the essential singularities is crucial for constructing efficient approximations of the function.

For that reason we develop a new version of the adaptive mechanism for detecting singularities. In this mechanism, the divided differences are computed only in the preliminary nonadaptive step for the uniform mesh of size $\mathcal{O}(n^{-1})$. Then a special extrapolation technique is applied to approximate an essential singularity (if it exists) with enough accuracy. This technique uses only $\mathcal{O}(\ln n)$ adaptively chosen points. Moreover, for $\delta = 0$, which corresponds to exact information, it is even simpler than the detection procedure of [15,16].

The improved detection mechanism allows to construct an adaptive algorithm using n function evaluations with precision δ that possesses the desired error properties in the worst case setting. In particular, its worst case error is proportional to $\max(\delta, n^{-(r+\varrho)})$, in the class of functions with uniformly bounded the Hölder coefficients and the discontinuity jumps $|f(s_f^+) - f(s_f^-)|$. For this result to hold the use of adaption, $p < \infty$, and uniform boundedness of the Hölder coefficients and the discontinuity jumps are crucial. Moreover, if we restrict the class even further to continuous functions, then the same worst case result can be achieved also for $p = \infty$ using no more than $(r-1)_+$ adaptive points.

The results of this paper can also be interpreted as follows. In order to obtain an ε-approximation in the worst case setting for the classes described above it is necessary and sufficient to use $n = \Omega(\varepsilon^{-1/(r+\varrho)})$ function evaluations with precision $\delta = \mathcal{O}(\varepsilon)$.

In Section 2 we formally define our approximation problem and provide lower bounds on the minimal error. In Section 3 we show some auxiliary results. The adaptive algorithm is presented and its cost and error analysis is provided correspondingly in Sections 4 and 5. Section 6 contains results of some numerical experiments.