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Newton's method on generalized Banach spaces

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ABSTRACT

We present a weaker convergence analysis of Newton's method than in Kantorovich and Akilov (1964), Meyer (1987), Potra and Ptak (1984), Rheinboldt (1978), Traub (1964) on a generalized Banach space setting to approximate a locally unique zero of an operator. This way we extend the applicability of Newton's method. Moreover, we obtain under the same conditions in the semilocal case weaker sufficient convergence criteria; tighter error bounds on the distances involved and an at least as precise information on the location of the solution. In the local case we obtain a larger radius of convergence and higher error estimates on the distances involved. Numerical examples illustrate the theoretical results.

1. Introduction

We present a semilocal convergence analysis for Newton's method on a generalized Banach space setting to approximate a zero of an operator. A generalized norm is defined to be an operator from a linear space into a partially order Banach space (to be precised in Section 2). The motivation for introducing generalized Banach spaces lies in the fact that this way the metric properties of the problem at hand can be analyzed more accurately. Moreover, convergence domains and estimates on the error distances involved are improved, when compared to the real norm theory [4,6,7,18].

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In the present study we expand the applicability of Newton's method using more precise majorizing sequences. In particular, we obtain the following advantages over the earlier mentioned studies using Newton's method.

Semilocal case:

(i) Weaker sufficient semilocal convergence criteria.
(ii) Tighter error bounds on the distances involved.
(iii) An at least as precise information on the location of the zero.

Local case:

(iv) An at least as large radius of convergence;
(v) More precise error estimates on the distances involved.

These advantages are obtained under the same computational cost using a combination of Lipschitz and center-Lipschitz condition.

The rest of the paper is organized as follows: Section 2 contains the basic concepts on generalized Banach spaces and auxiliary results on inequalities and fixed points. In Section 3 we present the semilocal and local convergence analysis of Newton's method. Finally, in the concluding Section 4, we present some examples.

2. Generalized Banach spaces

We present some standard concepts that are needed in what follows to make the paper as self contained as possible. More details on generalized Banach spaces can be found in [3–5,18], and the references there in.

Definition 2.1. A generalized Banach space is a triplet $(x, E, /\cdot/)$ such that

(i) X is a linear space over $\mathbb{R}\,(\mathbb{C})$.
(ii) $E = (E, K, \|\cdot\|)$ is a partially ordered Banach space, i.e. (ii_1) $(E, \|\cdot\|)$ is a real Banach space,
(ii_2) E is partially ordered by a closed convex cone K,
(iii_3) The norm $\|\cdot\|$ is monotone on K.
(iii) The operator $/\cdot/ : X \to K$ satisfies

$$/x/ = 0 \Leftrightarrow x = 0, \qquad /\theta x/ = |\theta|\,/x/\,,$$
$$/x + y/ \leq /x/ + /y/ \quad \text{for each } x, y \in X, \theta \in \mathbb{R}(\mathbb{C}).$$

(iv) X is a Banach space with respect to the induced norm $\|\cdot\|_i := \|\cdot\| \cdot /\cdot/$.

Remark 2.2. The operator $/\cdot/$ is called a generalized norm. In view of (iii) and (ii_3) $\|\cdot\|_i$, is a real norm. In the rest of this paper all topological concepts will be understood with respect to this norm.

Let $L\left(X^j, Y\right)$ stand for the space of j-linear symmetric and bounded operators from X^j to Y, where X and Y are Banach spaces. For X, Y partially ordered $L_+\left(X^j, Y\right)$ stands for the subset of monotone operators P such that

$$0 \leq a_i \leq b_i \Rightarrow P\left(a_1, \ldots, a_j\right) \leq P\left(b_1, \ldots, b_j\right). \tag{2.1}$$

Definition 2.3. The set of bounds for an operator $Q \in L(X, X)$ on a generalized Banach space $(X, E, /\cdot/)$ the set of bounds is defined to be:

$$B(Q) := \{P \in L_+(E, E)\,,\, /Qx/ \leq P\,/x/ \text{ for each } x \in X\}\,. \tag{2.2}$$

Let $D \subset X$ and $T : D \to D$ be an operator. If $x_0 \in D$ the sequence $\{x_n\}$ given by

$$x_{n+1} := T(x_n) = T^{n+1}(x_0) \tag{2.3}$$

is well defined. We write in case of convergence

$$T^{\infty}(x_0) := \lim\left(T^n(x_0)\right) = \lim_{n\to\infty} x_n. \tag{2.4}$$