Journal of Complexity 35 (2016) 102–123

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

Optimal recovery of integral operators and its applications

Vladyslav Babenko[a], Yuliya Babenko[b,*], Nataliia Parfinovych[a], Dmytro Skorokhodov[a]

[a] *Department of Mathematics and Mechanics, Dnepropetrovsk National University, Gagarina pr., 72, Dnepropetrovsk, 49010, Ukraine*

[b] *Department of Mathematics, Kennesaw State University, 1100 South Marietta Pkwy, MD # 9085, Marietta, GA, 30060, USA*

ARTICLE INFO

Article history:
Received 2 October 2015
Accepted 15 February 2016
Available online 27 February 2016

Keywords:
Optimal recovery
Information with error
Integral operators
Integral equations
Initial and boundary value problems

ABSTRACT

In this paper we present the solution to a problem of recovering a rather arbitrary integral operator based on incomplete information with error. We apply the main result to obtain optimal methods of recovery and compute the optimal error for the solutions to certain integral equations as well as boundary and initial value problems for various PDE's.

1. Introduction

Solutions to boundary (or initial) value problems for various partial differential equations require knowledge of a boundary (or initial) function. However, those functions are not often fully known and only partial information about them can be measured, e.g. values at some finite set of points, average values over small measurement intervals, values of N first consecutive Fourier coefficients, etc. Thus, it is very important to find an approximate solution based on available information on the boundary (or initial) function. Furthermore, it is also natural and important to develop methods that provide an optimal (in some sense) approximation to the true solution. These research questions have been explored under the theory of optimal recovery of functions and operators, which is a domain

* Corresponding author.
E-mail address: ybabenko@kennesaw.edu (Y. Babenko).

http://dx.doi.org/10.1016/j.jco.2016.02.004

of Approximation Theory and Information-based Complexity that started to develop in 1970s. More information on the development of the area can be found, for instance, in [21,29,22,12,16,26,23,13,32].

As for specific applications to recovering solutions of boundary and initial value problems, Magaril-Il'yaev, Osipenko, and co-authors (see, for instance, [17,24,18,30]) have considered the problem of optimal L_2-approximation of the solution to the Dirichlet problem for Laplace's and Poisson's equations in simple domains (disk, ball, annulus) based on the first N consecutive Fourier coefficients of the boundary function (possibly given with an error). In order to solve this problem they have used methods of Harmonic Analysis and general results from Optimization Theory.

In this paper we address related questions of optimal approximation of the solution to several types of integral equations, boundary and initial value problems for PDE's. We begin by solving a more general problem of recovering a rather arbitrary integral operator and sum of operators. We then present the optimal *method of recovery* as well as the *optimal error*. Next, we apply this general result to recover solutions to various boundary and initial value problems. Moreover, we present optimal methods of recovery of the solution to boundary-value problems based on this incomplete information with error. Naturally, the solution to the problem when information with error is used will also lead to the solution to the problem with exact information. In this paper we focus on considering the Volterra's and Fredholm's linear integral equations as well as boundary value problems for wave, heat, and Poisson's equations. Nevertheless, the developed method is more general and can be applied to other similar problems.

The paper is organized as follows. Section 2 contains necessary definitions and notation as well as the formulation and solution of the main problem. In Section 3, we solve the problem of optimal recovery of positive integral operators on classes of functions defined by moduli of continuity, based on information with an error about values of such functions at a fixed system of points. In Section 4, we use our general result from Section 3 to address optimal recovery problems for the solutions of Volterra and Fredholm integral equations of the second kind, systems of linear first order differential equations with constant coefficients, Poisson's equation, the heat and wave equations.

2. Statement and solution of the main problem

2.1. Definitions and notation

For $l, m \in \mathbb{N}$, we let $\left\{X_j\right\}_{j=1}^m$ be a collection of real linear spaces, $\{Y_i\}_{i=1}^l$ be a collection of real linear normed spaces, and $\left\{Z_j\right\}_{j=1}^m$ be a collection of real linear spaces. Set

$$\overline{X} := X_1 \times \cdots \times X_m, \qquad \overline{Y} := Y_1 \times \cdots \times Y_l, \qquad \overline{Z} := Z_1 \times \cdots \times Z_m.$$

We write elements of spaces $\overline{X}, \overline{Y}, \overline{Z}$ as column-vectors, *e.g.* $\bar{x} \in \overline{X}$ is a column-vector consisting of elements $x_1, x_2, \ldots, x_m$ with $x_j \in X_j$, $j = 1, \ldots, m$. This allows us to equip spaces $\overline{X}, \overline{Y}, \overline{Z}$ with natural coordinate-wise linear structure. In addition, in the space $\overline{Y}$ we introduce the norm

$$\|\bar{y}\|_{\overline{Y}} = \|\bar{y}\|_{\psi} := \psi\left(\|y_1\|_{Y_1}, \ldots, \|y_l\|_{Y_l}\right), \tag{1}$$

where ψ is an arbitrary norm in $\mathbb{R}^l$, monotone with respect to the natural partial order in $\mathbb{R}^l$.

By θ we denote zero of a linear space. It will be clear from the context what space is being discussed and, hence, we omit specifying it in the notation.

Next, for a collection of linear operators $A_{i,j} : X_j \to Y_i$, $i = 1, \ldots, l$, and $j = 1, \ldots, m$, with domains $D\left(A_{i,j}\right)$ we consider operator matrix

$$\overline{A} := \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{l,1} & A_{l,2} & \cdots & A_{l,m} \end{pmatrix}.$$