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Weyl and Bernstein numbers of embeddings of Sobolev spaces with dominating mixed smoothness

Van Kien Nguyen *

Friedrich-Schiller-University Jena, Ernst-Abbe-Platz 2, 07737 Jena, Germany
University of Transport and Communications, Lang Thuong, Dong Da, Hanoi, Viet Nam

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ABSTRACT

This paper is a continuation of work of the author and joint work with Winfried Sickel. Here we shall investigate the asymptotic behaviour of Weyl and Bernstein numbers of embeddings of Sobolev spaces with dominating mixed smoothness into Lebesgue spaces.

1. Introduction

Let $\Omega = (0, 1)^d$. The purpose of the present paper is to study the order of convergence of Weyl and Bernstein numbers of embeddings of Sobolev spaces of dominating mixed smoothness $S^t_{p_1}H(\Omega)$ into Lebesgue spaces $L_{p_2}(\Omega)$. Let us first recall the notions of Weyl and Bernstein numbers. Since the results on Weyl and Bernstein numbers are compared with the corresponding results of entropy numbers, we also give the definition of entropy numbers.

Definition 1.1. Let X, Y be Banach spaces and T be a continuous linear operator from X to Y, i.e., $T \in \mathcal{L}(X, Y)$.

(i) The nth Weyl number of T is defined as

$$x_n(T) := \sup\{a_n(TA) : A \in \mathcal{L}(\ell_2, X),\ \|A\| \le 1\}, \quad n \in \mathbb{N}.$$

* Correspondence to: Friedrich-Schiller-University Jena, Ernst-Abbe-Platz 2, 07737 Jena, Germany.
E-mail addresses: kien.nguyen@uni-jena.de, kiennv@utc.edu.vn.

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Here a_n is the nth approximation number which is defined as

$$a_n(T) := \inf\{\|T - A : X \to Y\| : A \in \mathcal{L}(X,Y),\ \mathrm{rank}(A) < n\}, \quad n \in \mathbb{N}.$$

(ii) The nth Bernstein number of T is defined as

$$b_n(T) = \sup_{L_n} \inf_{\substack{x\in L_n \\ x\neq 0}} \frac{\|Tx \mid Y\|}{\|x \mid X\|},$$

where the supremum is taken over all subspaces L_n of X with dimension n.

(iii) The nth (dyadic) entropy number of T is defined as

$$e_n(T) := \inf\{\varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{n-1} \text{ balls in } Y \text{ of radius } \varepsilon\},$$

where $B_X := \{x \in X : \|x|X\| \le 1\}$ denotes the closed unit ball of X.

Weyl numbers have been introduced by Pietsch [32]. The particular interest in Weyl numbers stems from the fact that they are the smallest known s-numbers satisfying the famous Weyl-type inequalities. Let $T : X \to X$ be a compact linear operator in a Banach space X and $\{\lambda_n(T)\}_{n=1}^{\infty}$ be the sequence of non-zero eigenvalues of T, ordered in the following way: each eigenvalue is repeated according to its algebraic multiplicity and $|\lambda_n(T)| \ge |\lambda_{n+1}(T)|$, $n \in \mathbb{N}$. Then

$$\Big(\prod_{k=1}^{2n-1} |\lambda_k(T)|\Big)^{1/(2n-1)} \le \sqrt{2e}\,\Big(\prod_{k=1}^{n} x_k(T)\Big)^{1/n}$$

holds for all $n \in \mathbb{N}$, in particular,

$$|\lambda_{2n-1}(T)| \le \sqrt{2e}\,\Big(\prod_{k=1}^{n} x_k(T)\Big)^{1/n}, \tag{1.1}$$

see Pietsch [32] and Carl, Hinrichs [8]. This inequality should be compared with the Carl–Triebel inequality which states

$$|\lambda_n(T)| \le \sqrt{2}\, e_n(T), \tag{1.2}$$

see Carl, Triebel [10] (see also [9,15]). In some situations, the inequality (1.1) is better than (1.2). Hence, it shows the importance of estimates of Weyl numbers in the study of eigenvalue distributions of compact operators. Pietsch [31,33] (in one dimension) and König [19–21] have used the behaviour of Weyl numbers $x_n(id : B^t_{p,q}(\Omega) \to L_s(\Omega))$ to estimate eigenvalues of the compact integral operator $T_K f(x) = \int_\Omega k(x,y) f(y) dy$ in a Lebesgue space on Ω, where the kernel $K(x,y)$ belongs to $B^{t_1}_{p_1,q_1}(\Omega, B^{t_2}_{p_2,q_2}(\Omega))$, $x, y \in \Omega$. Here $B^t_{p,q}(\Omega)$ denotes the isotropic Besov spaces defined on Ω. They showed that if $t_1, t_2 > 0$ and $t_1 + t_2 > d\big(\frac{1}{p_1} + \frac{1}{p_2} - 1\big)$ then the eigenvalues of T_K belong to a certain Lorentz sequence space. Based on the results in Theorem 2.1 and Proposition 2.2 we are able to control the eigenvalues of the operator $T : L_{p_2}(\Omega) \to L_{p_2}(\Omega)$ which can be decomposed $T = id \circ A$ where $A : L_{p_2}(\Omega) \to S^t_{p_1}H(\Omega)$ and $id : S^t_{p_1}H(\Omega) \to L_{p_2}(\Omega)$, $1 < p_1, p_2 < \infty$ and $t > (\frac{1}{p_1} - \frac{1}{p_2})_+$. By inequalities (1.1), (1.2) and properties of Weyl and entropy numbers we can show that

$$|\lambda_n(T)| \le C\,\|A\| \cdot \min\big\{e_n(id), x_n(id), e_n(id^*), x_n(id^*)\big\},$$

where id^* is dual operator of id, i.e., id^* is the embedding from $L_{p_2'}(\Omega)$ into $S^{-t}_{p_1'}H(\Omega)$. From lifting property of Sobolev spaces of dominating mixed smoothness we can deduce $\gamma_n(id^*) \asymp \gamma_n(id_1 : S^t_{p_2'}H(\Omega) \to L_{p_1'}(\Omega))$, here γ_n denotes either Weyl or entropy numbers. These are actually the results in Theorem 2.1 and Proposition 2.2. For more applications of Weyl numbers, we refer to [20,23,33].

The behaviour of Weyl numbers has been considered at various places since 1980, for example, Pietsch [30,32], Lubitz [23], König [20] and Caetano [5–7]. They studied Weyl numbers of embeddings $id : B^t_{p_1,q_1}(\Omega) \to L_{p_2}(\Omega)$. Recently, Zhang, Fang, Huang [51] and Gasiorowska, Skrzypczak [17] investigated the case of embeddings of weighted Besov spaces, defined on $\mathbb{R}^d$, into Lebesgue spaces,