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Estimates for the entropy numbers of embedding operators of function spaces on sets with tree-like structure: Some limiting cases

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ABSTRACT

In this paper we obtain order estimates for the entropy numbers of embedding operators of weighted Sobolev spaces into weighted Lebesgue spaces, as well as two-weighted summation operators on trees. Here, the parameters satisfy some critical conditions.

1. Introduction

In [40,44] order estimates for the entropy numbers $e_n(I : \hat{W}^r_{p,g}(\Omega) \to L_{q,v}(\Omega))$ of the embedding operator I of a weighted Sobolev space $\hat{W}^r_{p,g}(\Omega)$ on a John domain $\Omega \subset \mathbb{R}^d$ into a weighted Lebesgue space $L_{q,v}(\Omega)$ were obtained (all notation will be given in Section 4). In particular, it was proved that if $g(x) = \varphi_g(\mathrm{dist}(x, \Gamma))$, $v(x) = \varphi_v(\mathrm{dist}(x, \Gamma))$, where $\Gamma \subset \partial\Omega$ is an h-set with $h(t) = t^\theta |\log t|^\gamma$, $0 < \theta < d$, $\varphi_g(t) = t^{-\beta_g}|\log t|^{-\alpha_g}$, $\varphi_v(t) = t^{-\beta_v}|\log t|^{-\alpha_v}$, $1 < p < q < \infty$, $\beta_g + \beta_v = r + \frac{d}{q} - \frac{d}{p} > 0$, $\beta_v < \frac{d-\theta}{q}$, $0 < \alpha_g + \alpha_v \neq \frac{1}{p} - \frac{1}{q}$, then

$$e_n(I : \hat{W}^r_{p,g}(\Omega) \to L_{q,v}(\Omega)) \asymp \begin{cases} n^{\frac{1}{q}-\frac{1}{p}} (\log n)^{-\alpha_g - \alpha_v - \frac{1}{q} + \frac{1}{p}} & \text{if } \alpha_g + \alpha_v > \dfrac{1}{p} - \dfrac{1}{q}, \\ n^{-\alpha_g - \alpha_v} & \text{if } \alpha_g + \alpha_v < \dfrac{1}{p} - \dfrac{1}{q}. \end{cases}$$

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Here we consider the critical case $\alpha_g + \alpha_v = \frac{1}{p} - \frac{1}{q}$ and prove that

$$e_n(I : \hat{\mathcal{W}}^r_{p,g}(\Omega) \to L_{q,v}(\Omega)) \asymp n^{\frac{1}{q}-\frac{1}{p}}.$$

Recall the definition of the entropy numbers (see, e.g., [7,10,34]).

Definition 1. Let X, Y be normed spaces, and let $T : X \to Y$ be a linear continuous operator. The entropy numbers of T are defined by

$$e_k(T) = \inf\left\{\varepsilon > 0 : \exists y_1, \dots, y_{2^{k-1}} \in Y : T(B_X) \subset \cup_{i=1}^{2^{k-1}}(y_i + \varepsilon B_Y)\right\}, \quad k \in \mathbb{N}.$$

The problem of estimating entropy numbers or ε-entropy was intensively studied by different authors [3,8–30,32,33,35,36,38]. For details, see [40].

Triebel [38] and Mieth [32,33] obtained estimates for the entropy numbers of embedding operators of the weighted Sobolev space $\mathcal{W}^r_{p,g}(B)$ into $L_p(B)$; here B is a ball and the weight g has singularity at the origin.

The problem of estimating the entropy numbers of two-weighted summation operators

$$S_{u,w,\mathcal{T}}f(\xi) = w(\xi)\sum_{\xi' \leqslant \xi} u(\xi')f(\xi')$$

on a tree $\mathcal{T}$ was investigated in [26,28,29] and in [40]. In particular, if $2 \leqslant p < \infty$, $q = \infty$, $\mathcal{T}$ is a binary tree, $w(\xi) \equiv 1$, $u(\xi) = (|\xi| + 1)^{-\gamma/p'}$ (here $|\xi|$ is the distance between the vertex ξ and the root of $\mathcal{T}$, $p' := \frac{p}{p-1}$), $\gamma > 1$, then from results of Lifshits and Linde [26,28,29] and duality results for the entropy numbers [4] it follows that

$$e_n(S_{u,w,\mathcal{T}} : l_p(\mathcal{T}) \to l_\infty(\mathcal{T})) \asymp \begin{cases} n^{-\frac{1}{p}}(\log n)^{1-\frac{\gamma}{p'}}, & \gamma > p', \\ n^{-\frac{\gamma-1}{p'}}, & \gamma \leqslant p'; \end{cases}$$

the critical case $\gamma = p'$ was studied in [26] (for $p = 2$) and in [29]. In [40] the case $1 < p \leqslant \infty$, $1 \leqslant q < \infty$ was considered. In particular, it was proved that if $\mathcal{T}$ is a binary tree, $1 < p < q < \infty$, $u(\xi) = 2^{\beta|\xi|}(|\xi| + 1)^{-\alpha_u}$, $w(\xi) = 2^{-\beta|\xi|}(|\xi| + 1)^{-\alpha_w}$, $\beta > \frac{1}{q}$, $0 < \alpha_u + \alpha_w \neq \frac{1}{p} - \frac{1}{q}$, then

$$e_n(S_{u,w,\mathcal{T}} : l_p(\mathcal{T}) \to l_q(\mathcal{T})) \asymp \begin{cases} n^{\frac{1}{q}-\frac{1}{p}}(\log n)^{-\alpha_u-\alpha_w-\frac{1}{q}+\frac{1}{p}}, & \alpha_u + \alpha_w > \dfrac{1}{p} - \dfrac{1}{q}, \\ n^{-\alpha_u-\alpha_w}, & \alpha_u + \alpha_w < \dfrac{1}{p} - \dfrac{1}{q}. \end{cases}$$

Here we consider the critical case $\alpha_u + \alpha_w = \frac{1}{p} - \frac{1}{q}$ (see Section 5) and prove that

$$e_n(S_{u,w,\mathcal{T}} : l_p(\mathcal{T}) \to l_q(\mathcal{T})) \asymp n^{\frac{1}{q}-\frac{1}{p}}.$$

This paper is organized as follows. In Section 2 we obtain a general result about upper estimates for the entropy numbers of embedding operators of function spaces on a set with tree-like structure. The properties of such spaces are almost the same as those of function spaces defined in [40,46] (see Assumptions 1–3), but there are some differences. In particular, here we suppose that (2) holds. This condition cannot be directly derived from the known results for weighted Sobolev and Lebesgue spaces. Therefore, we first consider some particular cases of function spaces satisfying Assumptions A–C or A, B and D (see Section 3) and prove that these spaces satisfy Assumptions 1–3. In Section 4 we obtain order estimates for the entropy numbers of embedding operators of weighted Sobolev spaces; to this end, we prove that under given conditions on weights Assumptions A–C or A, B and D hold. In Section 5 we obtain order estimates for the entropy numbers of two-weighted summation operators on a tree.