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Tent-transformed lattice rules for integration and approximation of multivariate non-periodic functions

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We develop algorithms for multivariate integration and approximation in the weighted half-period cosine space of smooth non-periodic functions. We use specially constructed tent-transformed rank-1 lattice points as cubature nodes for integration and as sampling points for approximation. For both integration and approximation, we study the connection between the worst-case errors of our algorithms in the cosine space and the worst-case errors of some related algorithms in the well-known weighted Korobov space of smooth periodic functions. By exploiting this connection, we are able to obtain constructive worst-case error bounds with good convergence rates for the cosine space.

1. Introduction

In this paper we consider multivariate integration and approximation in the *weighted half-period cosine space*. We use *tent-transformed rank-1 lattice points* as cubature nodes for integration and as sampling points for approximation. Lattice rules have been widely studied in the context of

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multivariate integration, see [5,24,28]. Rank-1 lattice point sets are completely described by the number of points n and an integer *generating vector* $\boldsymbol{z}$, which can be constructed by an algorithm that searches for its elements *component by component*, see e.g., [6,15,25,26,29–31].

We will focus on the non-periodic setting and, as in [7], we will use the half-period cosine space spanned by the cosine series. Cosine series are used for the expansion of non-periodic functions in the d-dimensional unit cube. They are the eigenfunctions of the Laplace differential operator with homogeneous Neumann boundary conditions. The half-period cosine functions form a set of orthonormal basis functions of $L_2([0, 1])$ and are given by

$$\phi_0(x) = 1, \quad \text{and} \quad \phi_k(x) = \sqrt{2}\cos(\pi kx) \quad \text{for } k \in \mathbb{N}.$$

In d dimensions we will use the tensor products of these functions

$$\phi_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^{d} \phi_{k_j}(x_j) = \sqrt{2}^{|\boldsymbol{k}|_0} \prod_{j=1}^{d} \cos(\pi k_j x_j), \tag{1}$$

where we denote by $|\boldsymbol{k}|_0$ the number of non-zero elements of $\boldsymbol{k} \in \mathbb{Z}_+^d$, with

$$\mathbb{Z}_+ := \{0, 1, 2, \ldots\}.$$

The cosine series expansion of a d-variate function $f \in L_2([0, 1]^d)$ converges to f in the L_2 norm. Additionally, if f is continuously differentiable, we have uniform convergence and f can be expressed as a cosine series expansion as follows, see [1,10]:

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}_+^d} \hat{f}(\boldsymbol{k})\, \phi_{\boldsymbol{k}}(\boldsymbol{x}),$$

where $\hat{f}(\boldsymbol{k})$ are the cosine coefficients of f and are obtained as follows

$$\hat{f}(\boldsymbol{k}) = \int_{[0,1]^d} f(\boldsymbol{x})\, \phi_{\boldsymbol{k}}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

Cosine series overcome the well-known Gibbs phenomenon, which traditional Fourier series face in the expansion of non-periodic functions. Cosine series and the spectral methods using them have been studied in depth in [1,10] and their successors.

The precise definition of the weighted half-period cosine space will be presented in Section 2. For now we mention only that there is a parameter $\alpha > 1/2$ which characterizes the smoothness of the space by controlling the decay of the cosine coefficients, and there is a sequence of weights $1 \geq \gamma_1 \geq \gamma_2 > \cdots > 0$ which models the relative importance between successive coordinate directions.

We will first look at the problem of multivariate integration, where we will use tent-transformed lattice points as cubature nodes. Lattice rules have traditionally been used for the integration of smooth periodic functions. In the Korobov space of smooth periodic functions, it is known that lattice rules with well-chosen generating vectors can achieve the (almost optimal) rate of convergence of $\mathcal{O}(n^{-\alpha+\delta})$, for any $\delta > 0$, see, e.g., [6,15]. Moreover, the result for the case $\alpha = 1$ can be used to prove that randomly-shifted lattice rules can achieve the (almost optimal) rate of convergence of $\mathcal{O}(n^{-1+\delta})$ for $\delta > 0$ in the Sobolev spaces of non-periodic functions of dominating mixed smoothness 1. Tent-transformed lattice rules were first used to integrate non-periodic functions in [8], in the setting of unanchored Sobolev spaces of dominating mixed smoothness 1 and 2. It was shown there that when the lattice points are first randomly shifted and then tent-transformed (called bakers' transform in [8]), they can achieve the convergence rates of $\mathcal{O}(n^{-1+\delta})$ and $\mathcal{O}(n^{-2+\delta})$, $\delta > 0$, in the Sobolev spaces of smoothness 1 and 2, respectively.

In [7], tent-transformed lattice points were studied for integration in the weighted half-period cosine space without random shifting. It was claimed there that the worst-case error in the cosine space for a tent-transformed lattice rule is the same as the worst-case error in the weighted Korobov space of smooth periodic functions using lattice rules, given the same set of weights γ_j and the smoothness parameter α. The argument was based on achieving equality in a Cauchy–Schwarz type