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On the approximate calculation of multiple integrals

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ABSTRACT

When approximately calculating integrals of high dimension with the Monte Carlo method, one uses fewer values of the integrand than when calculating with the help of classical deterministic quadrature formulas.

However, the error estimation for the Monte Carlo method does not depend on the smoothness of the integrand. This suggests that it is possible to obtain methods that give a better order of convergence in case of smooth functions.

1. Lower estimation of the possible error

We show a lower bound for the maximum error of evaluating integrals for functions of a given smoothness.

Consider the following problem: suppose we have the class of all functions $f_j(n)$ $(n = 1, 2, \dots, 2N)$ that take the values ± 1, and one wants to evaluate the arithmetic mean of a function from this class

$$S_j = \frac{\sum_{n=1}^{2N} f_j(n)}{2N}.$$

We have the following lemma.

Lemma 1. *If a deterministic method of approximately evaluating the arithmetic mean uses information about the function values at no more than N points, then there is a function from the class for which the difference between the computed arithmetic mean and the true value is at least* $1/2$.

Indeed, suppose that the arithmetic mean was evaluated using information about values of f at points $i_1, i_2, \dots, i_m$ $(m \le N)$, and that $f(i_k) = \alpha_k$ $(k = 1, \dots, m)$. Consider functions $f_t(n)$ and $f_\tau(n)$ such that $f_t(i_k) = f_\tau(i_k) = \alpha_k$ $(k = 1, \dots, m)$ and $f_t(n) = -f_\tau(n) = 1$ at the other points.

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Since $|S_t - S_\tau| \geq 1$ and the calculated approximate values $\overline{S}_t$ and $\overline{S}_\tau$ of the arithmetic means are the same, then

$$\max(|S_t - \overline{S}_t|,\ |S_\tau - \overline{S}_\tau|) \geq \frac{1}{2}.$$

Definition 1. A method of evaluating an integral (arithmetic mean) is called non-deterministic if the coordinates of each successive knot are random functions that depend on the integration domain and all values obtained previously in the computational process (coordinates of the knots, function values and its derivatives, the number of knots, etc.), and the approximate value of the integral (arithmetic mean) is a random function depending on all the obtained values.

This definition applies to the Monte Carlo method, the method proposed in [4], methods described in Sections 2 and 4 of the present paper, and many others.

Lemma 2. *For any nondeterministic method of evaluating the arithmetic mean that uses information about function values at no more than N points, there is a function in the given class for which the average of the absolute error is at least $C_0 N^{-1/2}$ where C_0 is an absolute constant.*

We will say that an event K_γ,

$$\gamma = \begin{pmatrix} \alpha_1, \dots, \alpha_{m_\gamma} \\ \beta_1, \dots, \beta_{m_\gamma} \end{pmatrix},$$

$m_\gamma \leq N$, $\alpha_k = \pm 1$, took place in the process of evaluating the arithmetic mean, if information about values of the function at points $\beta_1, \dots, \beta_{m_\gamma}$ was used, and

$$f(\beta_k) = \alpha_k \quad (k = 1, \dots, m_\gamma).$$

Since the approximate value of the arithmetic mean (integral) can be written using a finite number of binary digits, it can take no more than countably many values I_s $(s = 1, 2, \dots)$.

Denote by π_γ^j the probability of the event K_γ in case the integral is evaluated for the function f_j. By σ_s^γ we denote the probability that the approximate evaluation of the integral equals I_s under the condition that the event K_γ took place.

We associate with each function f_j from the considered class the probability $p_j = 2^{-2N}$; it is clear that

$$\sum_j p_j = 1.$$

Set

$$\rho_j = \sum_\gamma \pi_\gamma^j \left(\sum_s \sigma_s^\gamma |I_s - S_j| \right), \tag{1}$$

$$\rho = \sum_j \rho_j p_j. \tag{2}$$

Obviously, ρ_j is the average of the absolute value of the error when calculating the integral of the function f_j. Plugging (1) into (2) we have

$$\rho = \sum_\gamma \left(\sum_j \pi_\gamma^j p_j \right) \left(\sum_s \sigma_s^\gamma q_{\gamma,s} \right), \tag{3}$$

where

$$q_{\gamma,s} = \frac{\sum_j \pi_\gamma^j p_j |I_s - S_j|}{\sum_j \pi_\gamma^j p_j}.$$