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Analysis of discrete least squares on multivariate polynomial spaces with evaluations at low-discrepancy point sets

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A B S T R A C T

We analyze the stability and accuracy of discrete least squares on multivariate polynomial spaces to approximate a given function depending on a multivariate random variable uniformly distributed on a hypercube. The polynomial approximation is calculated starting from pointwise noise-free evaluations of the target function at low-discrepancy point sets. We prove that the discrete least-squares approximation, in a multivariate anisotropic tensor product polynomial space and with evaluations at low-discrepancy point sets, is stable and accurate under the condition that the number of evaluations is proportional to the square of the dimension of the polynomial space, up to logarithmic factors. This result is analogous to those obtained in Cohen et al. (2013), Migliorati et al. (2014), Migliorati (2013) and Chkifa et al. (in press) for discrete least squares with random point sets, however it holds with certainty instead of just with high probability. The result is further generalized to arbitrary polynomial spaces associated with downward closed multi-index sets, but with a more demanding (and probably nonoptimal) proportionality between the number of evaluation points and the dimension of the polynomial space.

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1. Introduction

In recent years, an increasing interest has been dedicated to the various fields of applied mathematics graviting around the issue of uncertain knowledge of data in computational models. The uncertainty can be treated by means of random variables distributed according to a given or unknown probability distribution. In the applications, the presence of multiple sources of uncertainties demands that a large number of random variables be employed. Therefore, the underlying challenge is the approximation of target quantities of interest which functionally depend on a large number of random variables. Starting from the classical Monte Carlo method, *i.e.* with random sampling points, several approaches have been proposed. When the functional dependences on the random variables are smooth, polynomial approximation techniques [18] such as stochastic Galerkin [2], stochastic collocation on sparse grids [5] and discrete least squares with random evaluations [7,22,19,6] have been proposed as an efficient approximation tool. Another approach is the quasi-Monte Carlo method [24,28,10], which relies on the careful development of specific sets of deterministic quadrature points, so-called low-discrepancy points, to approximate multidimensional integrals. The combination of random and deterministic points has proven advantageous as well.

In recent works, it has been proven that univariate discrete least squares on polynomial spaces with *random* evaluations uniformly distributed on an interval are stable and optimally convergent in expectation [7] and in probability [22], under the condition that the number of evaluations is proportional to the square of the dimension of the polynomial space. The analysis has been extended to the multivariate case in [6], for any dimension of the random variable, for polynomial spaces associated with any arbitrary downward closed multi-index set, for the uniform and Chebyshev densities. The same analysis can be extended to any tensorized densities on a hypercube in the beta family using the results proven in [20].

In the present work we focus only on the case of uniform density, and we analyze discrete least squares on multivariate polynomial spaces with evaluations at *low-discrepancy* point sets. We prove in Theorem 9 that, in multivariate anisotropic tensor product polynomial spaces and using low-discrepancy point sets, the discrete least-squares approximation of any uniformly continuous function is stable and accurate, when the number of evaluation points is proportional to the square of the dimension of the polynomial space (up to logarithmic factors). As in [6], accurate means that the error of the discrete least-squares projection in the L^2 norm is comparable with the best approximation error in the L^∞ norm. Therefore, with anisotropic tensor product spaces, the use of low-discrepancy point sets leads to analogous theoretical results as those with random points proven in [6]. The results with low-discrepancy points hold with certainty, whereas the results with random points only hold with high probability or in expectation. A closer look to the logarithmic factors reveals that in the low-discrepancy case the stability condition contains a logarithmic dependence which worsens as the dimension increases, whereas the same logarithmic dependence is dimension-free in the random case.

In the multivariate case, when the polynomial space differs from the anisotropic tensor product the quadratic growth worsens: in any case we have proven the stability and accuracy of discrete least squares in any polynomial space associated with arbitrary downward closed multi-index sets, if the number of evaluation points is proportional to the quartic power of the dimension of the polynomial space. Notice that this is a sufficient but not necessary condition. An analogous quartic proportionality can be proven using probabilistic estimates for the star discrepancy of random points independent and uniformly distributed.

A relevant quantity in our analysis is the superposition of star discrepancies of low-order projections of point sets, which has proven to be related to the convergence of quasi-Monte Carlo and to tractability issues, see [27,31] and references therein.

Recently, in [33] an analysis of discrete least squares with deterministic points has been presented in the case of the Chebyshev density, however, using techniques quite different than those used here. The authors prove stability and accuracy under the condition that the number of points scales as the square of the dimension of the polynomial space associated with any downward closed multi-index set, with the proportionality constant depending on the number of components of the multivariate random variable.