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Exponential convergence-tractability of general linear problems in the average case setting☆

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ABSTRACT

We study d-variate general linear problems defined over Hilbert spaces in the average case setting. We consider algorithms that use finitely many evaluations of arbitrary linear functionals. We obtain the necessary and sufficient conditions for Exponential Convergence-(Strong) Polynomial Tractability, Quasi-Polynomial Tractability and (Uniform) Weak Tractability. All results are in terms of the eigenvalues of the corresponding covariance operators.

1. Introduction

Multivariate computational problems are defined on classes of functions with d variables. In many important problems d is large or even huge. Traditionally the complexity of multivariate problems was studied with respect to the accuracy ε only. This ignores the dependence of the complexity on d. Tractability of multivariate continuous problems studies the complexity with respect to both ε^{-1} and d. The first result on this subject was proposed by Henryk Woźniakowski in 1994 (see [16]). Since then a huge number of results emerged on this topic. Tractability of multivariate problems has been a very active research area: see [9–11] and the references therein. Tractability of multivariate problems has been studied in different settings including the worst case, the probabilistic case and the average case setting. In this paper we limit ourselves to the average case setting.

Exponential convergence-tractability (it is abbreviated as EC-tractability in the following for simplicity) studies the complexity with respect to the number of bits of accuracy. In this vein of research, multivariate integration and approximation have been studied over Korobov spaces with

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exponentially fast decaying Fourier coefficients, see [2,1,5,6]. The EC-tractability notions were already introduced in [2,1,5] but the corresponding nomenclature was introduced later in [6]. Similar notions of tractability have been introduced by Papageorgiou and Petras [13] which they call "polylog-tractability" and "$\ln^{\kappa}$-weak tractability". We also noticed that tractability in the context of exponential error convergence is also addressed in the third volume of the books of Novak and Woźniakowski [11]. From above papers we know that EC-tractability studies the same notions of tractability in terms of $(\ln_+ \varepsilon^{-1}, d)$ (we will denote $\ln_+ x = 1 + \ln x$ in the following for simplicity) instead of (ε^{-1}, d). Similarly to the EC-tractability concepts in the worst case setting (see [6]), we list the different EC-tractability concepts in the average case setting below.

- Exponential Convergence-Weak Tractability if $n(\varepsilon, d)$ is not exponential in d and $\ln_+ \varepsilon^{-1}$.
- Exponential Convergence-Polynomial Tractability if $n(\varepsilon, d)$ is of order $d^q(\ln_+ \varepsilon^{-1})^p$.
- Exponential Convergence-Strong Polynomial Tractability if $n(\varepsilon, d)$ is of order $(\ln_+ \varepsilon^{-1})^p$.

Here the definition of $n(\varepsilon, d)$ can be found in Section 2. The bounds above hold for all $d \in \mathbb{N}$ and all $\varepsilon \in (0, 1]$ with the parameters q, p and the pre-factors independent of d and ε.

In the worst case setting, Exponential Convergence-Weak, Polynomial and Strong Polynomial Tractability for multivariate integration and approximation in weighted Korobov spaces with exponentially fast decaying Fourier coefficients were studied in the papers [2,1,5,6]. Recently, A. Papageorgiou and I. Petras [13] gave the necessary and sufficient conditions for "polylog tractability" which is the same concept as Exponential Convergence-Polynomial Tractability. Besides, A. Papageorgiou and I. Petras [13] gave the concept of $\ln^{\kappa}$-weak tractability and showed necessary and sufficient conditions for general linear problems and tensor product problems. In this paper we will consider the corresponding EC-tractability in the average case setting and study general linear multivariate problems defined over Hilbert spaces.

The concept of quasi-polynomial tractability in the traditional sense was introduced recently in [3]. Similar investigations can be found in [4,7,8,17,18]. Similarly to [3], we will give the concept of Exponential Convergence-Quasi-Polynomial Tractability and study general linear multivariate problems defined over Hilbert spaces.

The concept of uniform weak tractability in the traditional sense was introduced recently in [14], which mainly discusses linear tensor product problems. Similar investigations can be found in [15,18]. Similarly to [14], we will give the concept of Exponential Convergence-Uniform Weak Tractability and study general linear multivariate problems defined over Hilbert spaces.

The paper is organized as follows. Section 2 contains some basic concepts and results. In Section 3, we study Exponential Convergence-Weak Tractability. In Section 4, we study Exponential Convergence-Uniform Weak Tractability. In Section 5, we study Exponential Convergence-Quasi-Polynomial Tractability. In Section 6, we study Exponential Convergence-(Strong) Polynomial Tractability. In all sections we give the corresponding necessary and sufficient conditions which are in terms of the eigenvalues of its covariance operators. At last we give the proof of some results in the Appendix.

2. Some concepts

Let F_d be a Banach space of d-variate real functions defined on a Lebesgue measurable set $D_d \subset \mathbb{R}^d$. The space F_d is equipped with a zero-mean Gaussian measure μ_d defined on Borel sets of F_d. We denote by $C_{\mu_d} : F_d^* \to F_d$ (where and in the following F_d^* denotes the dual space of F_d) the covariance operator of μ_d, e.g., see [9, Appendix B] for its definition. Let H_d be a Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle_{H_d}$ and $\| \cdot \|_{H_d}$, respectively.

We want to approximate a continuous linear operator

$$S_d : F_d \to H_d.$$

Let $\nu_d = \mu_d S_d^{-1}$ be the induced measure. Then ν_d is a zero-mean Gaussian measure on the Borel sets of H_d with covariance operator $C_{\nu_d} : H_d \to H_d$ given by

$$C_{\nu_d} = S_d C_{\mu_d} S_d^*,$$