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Accessibility of solutions of operator equations by Newton-like methods

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ABSTRACT

The concept of a majorizing sequence introduced and applied by Rheinboldt in 1968 is taken up to develop a convergence theory of the Picard iteration $x_{n+1} = G(x_n)$ for each $n \geq 0$ for fixed points of an iteration mapping $G : D_0 \subset X \to X$ in a complete metric space X satisfying iterated contraction-like condition:

$$d(G(y), G(x)) \leq \psi(d(y, x), d(y, x_0), d(x, x_0))d(y, x)$$

for all $x \in D_0$ with $y = G(x) \in D_0$, where $x_0 \in D_0$ and $\psi \in \Phi(\mathcal{J}^3)$. Here $\mathcal{J}^3$ is a suitable set of $(\mathbb{R}^+)^3$ to be defined in Section 2. We study the region of accessibility of fixed points of G by the Picard iteration $u_{n+1} = G(u_n)$, where the starting point $u_0 \in D_0$ is not necessarily x_0. Our convergence theory is applied to the Newton-like iterations in Banach spaces under the center Lipschitz condition $\|F'_x - F'_{x_0}\| \leq \omega(\|x - x_0\|)$ for a given point $x_0 \in D_0$. Our results extend and improve the previous ones in the sense of the center Lipschitz condition and the region of accessibility of solutions. We apply our results to solve the nonlinear Fredholm operator equations of second kind.

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1. Introduction

Consider the nonlinear operator equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet differentiable operator defined on an open domain D of a Banach space X with values in a Banach space Y. We denote Fréchet-derivative of operator F at $x \in D$ by F'_x.

The problems of differential and integral equations, differential inequalities, optimization problems, variational problems, fixed points and many others can be formulated in terms of finding the solution of the nonlinear operator equation (1.1) (see [3,2,8–10,18,26]).

One of the most widely used numerical techniques for solving the operator equation (1.1) is Newton's method defined by the recursive formula

$$x_{n+1} = S(x_n) \tag{1.2}$$

for all $n \in \mathbb{N}_0$ with x_0 given and assuming that ${F'_{x_n}}^{-1}$ exists for all $n \in \mathbb{N}_0$, where $S(x) = x - {F'_x}^{-1}F(x)$. Note that Newton's method (1.2) is a successive substitution of the operator S. It is well known that the problem solving the operator equation (1.1) reduces to the problem of finding a fixed point of the operator S.

In order to avoid the use of the inverse of the derivative of the operator F involved in S, some authors (see, for example, [6,4,3,8,18,26]) studied the Newton-like method given by

$$x_{n+1} = G(x_n) \tag{1.3}$$

for all $n \in \mathbb{N}_0$, where $G(x) = x - {T_x}^{-1}F(x)$. Here $B(X, Y)$ denotes the space of bounded linear operators from X to Y and $T_x \in B(X, Y)$ is an approximation to the Fréchet-derivative F'_x of the operator F at $x \in D$ [2]. The semilocal convergence analysis for Newton-like method (1.3) under various Lipschitz-type assumptions can be found in [3,2,8,24,25], and the references therein.

Recently, Appell et al. [1], Ezquerro and Hernández [11] and Proinov [21] proved semilocal convergence results for the special case $T_x = F'_x$ for all $x \in D$ by using the following general condition:

$$\|F'_x - F'_y\| \le \omega(\|x - y\|), \tag{1.4}$$

where ω is a nondecreasing and non-negative function on $\mathbb{R}^+$. They considered a function $h : [0, 1] \to \mathbb{R}^+$ such that

$$\omega(st) \le h(s)\omega(t) \tag{1.5}$$

for all $s \in [0, 1]$ and $t \in \mathbb{R}^+$. Such a function h always exists [21].

It is interesting to consider the following condition: there exists an $\ell_0 \ge 0$ such that

$$\|T_x - T_{x_0}\| \le \omega_0(\|x - x_0\|) + \ell_0 \tag{1.6}$$

for all $x \in D$ for convergence analysis of Newton-like method (1.3). For $\ell_0 = 0$ and $T_x = F'_x$, the condition (1.6) reduces to

$$\|F'_x - F'_{x_0}\| \le \omega_0(\|x - x_0\|). \tag{1.7}$$

Clearly, (1.7) is a weaker assumption than (1.4) (see [13]).

On the other hand, a solution x^* of Eq. (1.1) is said to be *accessible* from x_0 if the sequence $\{x_n\}$ generated from x_0 by Newton's method (1.2) converges to x^*. The set of points x_0 for which the sequence $\{x_n\}$ defined by (1.2) converges to x^* is called the *region of accessibility* of the solution x^* of Eq. (1.1).

In case of Newton-like method (1.3), the operator G defined by $G(x) = x - {T_x}^{-1}F(x)$ is not necessarily contraction. Therefore, the characterization of the region of accessibility of solutions of (1.1) by the Newton-like method (1.3) is interesting and difficult, even in the real scalar case (see [22]). In [12], Gutiérrez et al. studied the region of accessibility of solutions of (1.1) by Newton's method (1.2) in Banach spaces in terms of degree of the logarithmic convexity.