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Constructing good higher order polynomial lattice rules with modulus of reduced degree

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ABSTRACT

In this paper we investigate multivariate integration in weighted unanchored Sobolev spaces of smoothness of arbitrarily high order. As quadrature points we employ higher order polynomial lattice point sets over $\mathbb{F}_2$ which are randomly digitally shifted and then folded using the tent transformation. We first prove the existence of good higher order polynomial lattice rules which achieve the optimal rate of the mean square worst-case error, while reducing the required degree of modulus by half as compared to higher order polynomial lattice rules whose quadrature points are randomly digitally shifted but not folded using the tent transformation. Thus we are able to restrict the search space of generating vectors significantly. We then study the component-by-component construction as an explicit means of obtaining good higher order polynomial lattice rules. In a way analogous to Baldeaux et al. (2012), we show how to calculate the quality criterion efficiently and how to obtain the fast component-by-component construction using the fast Fourier transform. Our result generalizes the previous result shown by Cristea et al. (2007), in which the degree of smoothness is fixed at 2 and classical polynomial lattice rules are considered.

1. Introduction

In this paper we study multivariate integration of smooth functions defined over the s-dimensional unit cube $[0, 1)^s$,

$$I(f) = \int_{[0,1)^s} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

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Quasi-Monte Carlo (QMC) rules approximate $I(f)$ by

$$Q(f; P_N) = \frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n),$$

where a point set $P_N = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\} \subset [0,1)^s$ is chosen carefully so as to yield a small integration error.

Explicit constructions of point sets whose star-discrepancy is of order $N^{-1+\epsilon}$ for any $\epsilon > 0$ have been studied extensively. They are motivated by the so-called Koksma–Hlawka inequality, which states that the integration error $|I(f) - Q(f; P_N)|$ is bounded above by the variation of f in the sense of Hardy and Krause times the star-discrepancy of P_N. There are two prominent families for construction of good point sets: integration lattices [16,21] and digital nets and sequences [8,16]. Regarding explicit constructions of digital sequences, we refer to [8, Chapter 8] and [16, Chapter 4].

Polynomial lattice point sets, first proposed in [15], are a well-known special construction of digital nets based on rational functions over finite fields. The existence of low-discrepancy polynomial lattice point sets has been proven previously, see, e.g., [13,14]. QMC rules using polynomial lattice point sets as P_N are called polynomial lattice rules, which are defined as follows. We refer to [8,17] for more information on polynomial lattice rules.

For a prime b, let $\mathbb{F}_b := \{0, \dots, b-1\}$ be the finite field consisting of b elements. We denote by $\mathbb{F}_b[x]$ the set of all polynomials over $\mathbb{F}_b$ and by $\mathbb{F}_b((x^{-1}))$ the field of formal Laurent series over $\mathbb{F}_b$. Every element of $\mathbb{F}_b((x^{-1}))$ can be represented as

$$L = \sum_{l=w}^{\infty} t_l x^{-l},$$

for some integer w and $t_l \in \mathbb{F}_b$. For a positive integer m, we define the mapping v_m from $\mathbb{F}_b((x^{-1}))$ to the unit interval $[0, 1)$ by

$$v_m\left(\sum_{l=w}^{\infty} t_l x^{-l}\right) = \sum_{l=\max(1,w)}^{m} t_l b^{-l}.$$

We often identify an integer $n = n_0 + n_1 b + \cdots \in \mathbb{N}_0$ with a polynomial $n(x) = n_0 + n_1 x + \cdots \in \mathbb{F}_b[x]$, where we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $\mathbb{N}$ the set of positive integers. Using these notations, polynomial lattice rules are defined as follows.

Definition 1. For $m, s \in \mathbb{N}$, let $p \in \mathbb{F}_b[x]$ with $\deg(p) = m$ and let $\boldsymbol{q} = (q_1, \dots, q_s) \in (\mathbb{F}_b[x])^s$. A polynomial lattice rule over $\mathbb{F}_b$ is a QMC rule using a polynomial lattice point set $P_{b^m}(\boldsymbol{q}, p)$ consisting of b^m points that are given by

$$\boldsymbol{x}_n := \left(v_m\left(\frac{n(x)q_1(x)}{p(x)}\right), \dots, v_m\left(\frac{n(x)q_s(x)}{p(x)}\right)\right) \in [0,1)^s,$$

for $0 \le n < b^m$. The vector $\boldsymbol{q}$ and the polynomial p are respectively called the *generating vector* and the *modulus* of $P_{b^m}(\boldsymbol{q}, p)$.

The aim of this paper is to construct good point sets which can exploit the smoothness of an integrand so as to improve the convergence rate of the integration error. Two principles for constructing such point sets based on the concept of digital nets have been proposed so far. One is known as *higher order polynomial lattice rules* that are given by generalizing the definition of polynomial lattice rules, see, e.g., [2,3,7]. The other is based on a digit interlacing function applied to digital nets and sequences whose number of components is a multiple of the dimension, see [5,6]. Since we focus on the former principle in this paper, we only give the definition of higher order polynomial lattice rules in the following.