Journal of Complexity 31 (2015) 327–371

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

Proof techniques in quasi-Monte Carlo theory

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A R T I C L E   I N F O

Article history:
Available online 21 September 2014

Keywords:
Quasi-Monte Carlo
Numerical integration
Discrepancy
Worst-case error
Average-case error

A B S T R A C T

In this survey paper we discuss some tools and methods which are of use in quasi-Monte Carlo (QMC) theory. We group them in chapters on Numerical Analysis, Harmonic Analysis, Algebra and Number Theory, and Probability Theory. We do not provide a comprehensive survey of all tools, but focus on a few of them, including reproducing and covariance kernels, Littlewood–Paley theory, Riesz products, Minkowski's fundamental theorem, exponential sums, diophantine approximation, Hoeffding's inequality and empirical processes, as well as other tools. We illustrate the use of these methods in QMC using examples.

Contents

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http://dx.doi.org/10.1016/j.jco.2014.09.003
0885-064X/

1. Introduction

Quasi-Monte Carlo (QMC) rules are quadrature rules which can be used to approximate integrals defined on the s-dimensional unit cube $[0, 1]^s$

$$\int_{[0,1]^s} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \approx \frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n),$$

where $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$ are deterministically chosen quadrature points in $[0, 1)^s$. In QMC theory one is interested in a number of questions. Of importance is the integration error

$$\left|\int_{[0,1]^s} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n)\right|$$

and how it behaves as N and/or s increases. Various settings can be defined to analyze this error. For instance, one can consider the worst-case error: Here one uses a Banach space $(\mathcal{H}, \|\cdot\|)$ and considers

$$\mathrm{wce}(\mathcal{H}, \mathcal{P}) = \sup_{\substack{f\in\mathcal{H}\\ \|f\|\le 1}} \left|\int_{[0,1]^s} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n)\right|.$$

Particularly nice examples of such function spaces are so-called reproducing kernel Hilbert spaces. We review essential properties of reproducing kernel Hilbert spaces in Section 2. Other settings include the average case error: In this case one defines a probability measure μ on the function space $\mathcal{H}$ and then studies the expectation value of the integration error

$$\mathrm{ace}_p(\mathcal{H}, \mathcal{P}) = \left(\mathbb{E}\left|\int_{[0,1]^s} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n)\right|^p\right)^{1/p}.$$

Such an investigation can be carried out with the help of covariance kernels. There are a number of relations to reproducing kernels, which we also discuss in Section 2.

Covariance kernels also appear in stochastic processes, which themselves are important in applications in financial mathematics and partial differential equations (PDEs) with random coefficients, for instance. We discuss all these connections in the section on numerical analysis, Section 2, in which we also treat some further useful tools, like the use of bump functions to prove lower bounds and the