Journal of Complexity 31 (2015) 372–379

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

On permutation-invariance of limit theorems

I. Berkes [a,*], R. Tichy [b]

[a] Graz University of Technology, Institute of Statistics, Kopernikusgasse 24, 8010 Graz, Austria
[b] Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria

ARTICLE INFO

Article history:
Available online 21 June 2014

Keywords:
Lacunary series
Limit theorems
Permutation-invariance
Subsequence principle
Exchangeable sequences

ABSTRACT

By a classical principle of probability theory, sufficiently thin subsequences of general sequences of random variables behave like i.i.d. sequences. This observation not only explains the remarkable properties of lacunary trigonometric series, but also provides a powerful tool in many areas of analysis, such as the theory of orthogonal series and Banach space theory. In contrast to i.i.d. sequences, however, the probabilistic structure of lacunary sequences is not permutation-invariant and the analytic properties of such sequences can change after rearrangement. In a previous paper we showed that permutation-invariance of subsequences of the trigonometric system and related function systems is connected with Diophantine properties of the index sequence. In this paper we will study permutation-invariance of subsequences of general r.v. sequences.

1. Introduction

It is known that sufficiently thin subsequences of general r.v. sequences behave like i.i.d. sequences. For example, Révész [23] showed that if a sequence (X_n) of r.v.'s satisfies $\sup_n EX_n^2 < \infty$, then one can find a subsequence (X_{n_k}) and a r.v. $X \in L^2$ such that $\sum_{k=1}^{\infty} c_k(X_{n_k} - X)$ converges a.s. provided $\sum_{k=1}^{\infty} c_k^2 < \infty$. Under the same condition, Gaposhkin [13,14] and Chatterji [9,10] proved that there exists a subsequence (X_{n_k}) and r.v.'s $X \in L^2$, $Y \in L^1$, $Y \geq 0$ such that

$$\frac{1}{\sqrt{N}} \sum_{k \leq N} (X_{n_k} - X) \xrightarrow{d} N(0, Y) \tag{1.1}$$

* Corresponding author.
E-mail addresses: berkes@tugraz.at (I. Berkes), tichy@tugraz.at (R. Tichy).

http://dx.doi.org/10.1016/j.jco.2014.06.001
0885-064X/

and

$$\limsup_{N\to\infty} \frac{1}{\sqrt{2N \log\log N}} \sum_{k\le N} (X_{n_k} - X) = Y^{1/2} \quad \text{a.s.} \tag{1.2}$$

Here $N(0, Y)$ denotes the distribution of the r.v. $Y^{1/2}\zeta$, where ζ is a standard normal r.v. independent of Y. Komlós [18] showed that if $\sup_n E|X_n| < \infty$, then there exists a subsequence (X_{n_k}) and a r.v. $X \in L^1$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} X_{n_k} = X \quad \text{a.s.}$$

Chatterji [8] showed that if $\sup_n E|X_n|^p < \infty$ where $0 < p < 2$, then the conclusion of the previous theorem can be changed to

$$\lim_{N\to\infty} \frac{1}{N^{1/p}} \sum_{k=1}^{N} (X_{n_k} - X) = 0 \quad \text{a.s.}$$

for some $X \in L^p$. Note the randomization in all these examples: the role of the mean and variance of the subsequence (X_{n_k}) is played by random variables X, Y. For further limit theorems for subsequences of general r.v. sequences and for the history of the topic until 1966, see Gaposhkin [13].

Since the asymptotic properties of an i.i.d. sequence do not change if we permute its terms, it is natural to expect that limit theorems for lacunary subsequences of general r.v. sequences remain valid after any permutation of their terms. This is, however, not the case. By classical results of Salem and Zygmund [24,25] and Erdős and Gál [12], under the Hadamard gap condition

$$n_{k+1}/n_k \ge q > 1 \quad k = 1, 2, \dots \tag{1.3}$$

the sequence $(\sin 2\pi n_k x)$ satisfies

$$\frac{1}{\sqrt{N/2}} \sum_{k=1}^{N} \sin 2\pi n_k x \xrightarrow{d} N(0, 1) \tag{1.4}$$

and

$$\limsup_{N\to\infty} \frac{1}{\sqrt{N \log\log N}} \sum_{k=1}^{N} \sin 2\pi n_k x = 1 \quad \text{a.s.} \tag{1.5}$$

with respect to the probability space $((0, 1), \mathcal{B}, \mu)$, where μ denotes the Lebesgue measure. Erdős [11] and Takahashi [27] proved that (1.4), (1.5) remain valid under the weaker gap condition

$$n_{k+1}/n_k \ge 1 + ck^{-\alpha}, \quad k = 1, 2, \dots \tag{1.6}$$

for $0 < \alpha < 1/2$ and that for $\alpha = 1/2$ this becomes false. As it was shown in [2,3], under the Hadamard gap condition (1.3) the CLT (1.4) and the LIL (1.5) are permutation-invariant, i.e. they remain valid after any permutation of the sequence (n_k), but this generally fails under the gap condition (1.6). Similar results hold for lacunary sequences $f(n_k x)$, where f is a measurable function satisfying

$$f(x + 1) = f(x), \qquad \int_0^1 f(x)\, dx = 0, \qquad \int_0^1 f^2(x)\, dx < \infty. \tag{1.7}$$

In this case, assuming the Hadamard gap condition (1.3), the validity of the CLT

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(n_k x) \xrightarrow{d} N(0, \sigma^2) \tag{1.8}$$

and of its permuted version depend on the number of solutions of the Diophantine equation

$$an_k + bn_\ell = c, \quad 1 \le k, \ell \le N. \tag{1.9}$$