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On embeddings of weighted tensor product Hilbert spaces

Mario Hefter *, Klaus Ritter

Fachbereich Mathematik, Technische Universität Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany

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ABSTRACT

We study embeddings between tensor products of weighted reproducing kernel Hilbert spaces. The setting is based on a sequence of weights $\gamma_j > 0$ and sequences $1 + \gamma_j k$ and $1 + l_{\gamma_j}$ of reproducing kernels k such that $H(1 + \gamma_j k) = H(1 + l_{\gamma_j})$, in particular. We derive necessary and sufficient conditions for the norms on $\bigotimes_{j=1}^{s} H(1 + \gamma_j k)$ and $\bigotimes_{j=1}^{s} H(1 + l_{\gamma_j})$ to be equivalent uniformly in s. Furthermore, we study relaxed versions of uniform equivalence by modifying the sequence of weights, e.g., by constant factors, and by analyzing embeddings of the respective spaces. Likewise, we analyze the limiting case $s = \infty$.

1. Introduction

Embedding theorems deal with scales $(F_s^\alpha)_\alpha$ of function spaces on a common domain of dimension $s \in \mathbb{N}$, and one of the aims is to characterize those pairs of spaces F_s^α and F_s^β that permit a continuous embedding $i_s^{\alpha,\beta} : F_s^\alpha \hookrightarrow F_s^\beta$. A major application of embedding theorems in information-based complexity, approximation theory, and numerical mathematics is as follows: The existence of a continuous embedding $i_s^{\alpha,\beta}$ with norm $\|i_s^{\alpha,\beta}\|$ implies

$$e_n(F_s^\alpha) \leq \|i_s^{\alpha,\beta}\| \cdot e_n(F_s^\beta) \tag{1}$$

for many quantities e_n of interest, like nth minimal errors or n-widths.

* Corresponding author.
E-mail addresses: hefter@mathematik.uni-kl.de (M. Hefter), ritter@mathematik.uni-kl.de (K. Ritter).

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In the classical approach one studies the asymptotic behavior of $e_n(F_s^\alpha)$ as n tends to infinity with α and s being fixed, and the mere existence of continuous embeddings can already be exploited, since (1) yields $e_n(F_s^\alpha) = O\left(e_n(F_s^\beta)\right)$. In particular, if $F_s^\alpha = F_s^\beta$ as vector spaces with equivalent norms, then the sequences $(e_n(F_s^\alpha))_n$ and $(e_n(F_s^\beta))_n$ are weakly equivalent.

In contrast, tractability analysis studies the explicit dependence of $e_n(F_s^\alpha)$ on n and on the dimension s, which is crucial to fully understand the impact of a high dimension on the computational or approximation problem at hand. We refer to [7–9] for a comprehensive study and further references. Moreover, tractability analysis enables the study of the limiting case $s = \infty$, i.e., of computational or approximation problems for functions with infinitely many variables. Exploiting the existence of continuous embeddings $i_s^{\alpha,\beta}$ or the equivalence of norms on $F_s^\alpha = F_s^\beta$ for all $s \in \mathbb{N}$ in tractability analysis requires a tight control of the dependence of the norms of the respective embeddings on the dimension s.

In this paper we study scales of weighted tensor product Hilbert spaces, which are most often studied in tractability analysis. The starting point for the construction of these spaces is a reproducing kernel k on a domain $D \times D$ and a sequence $(\gamma_j)_{j\in\mathbb{N}}$ of positive weights. By assumption, the Hilbert space $H(1+k)$ with reproducing kernel $1+k$ is the orthogonal sum of the space $H(1)$ of constant functions and the space $H(k)$. The corresponding norm of $f \in H(1+\gamma_j k)$ is therefore given by

$$\|f\|_{1+\gamma_j k}^2 = P(f)^2 + \frac{1}{\gamma_j} \cdot \|f - P(f)\|_k^2,$$

where P denotes the orthogonal projection onto $H(1)$. The second scale is derived from an equivalent norm

$$\|f\|_{1+l_{\gamma_j}}^2 = \langle f, f\rangle + \frac{1}{\gamma_j} \cdot \|f - P(f)\|_k^2$$

on the same vector space $H = H(1+k) = H(1+\gamma_j k)$. By assumption, $\langle\cdot,\cdot\rangle$ is a properly normalized symmetric bilinear form on H that is continuous on $H(k)$, and actually we get a new reproducing kernel Hilbert space $H(1+l_{\gamma_j})$ in this way. As it turns out,

$$\forall f \in H : P(f) = \langle f, 1\rangle \tag{2}$$

forms a particular instance, since (2) is equivalent to $H(1)$ and $H(k)$ being orthogonal in the spaces $H(1+l_{\gamma_j})$, too.

The first result of this paper, Theorem 1, deals with embeddings

$$i_s^{\boldsymbol{\eta},\boldsymbol{\gamma}} : H(K_s^{\boldsymbol{\eta}}) \hookrightarrow H(L_s^{\boldsymbol{\gamma}})$$

between the tensor product spaces

$$H(K_s^{\boldsymbol{\eta}}) = \bigotimes_{j=1}^{s} H(1+\eta_j k) \quad \text{and} \quad H(L_s^{\boldsymbol{\gamma}}) = \bigotimes_{j=1}^{s} H(1+l_{\gamma_j})$$

of functions on D^s, where $\boldsymbol{\eta} = (\eta_j)_{j\in\mathbb{N}}$ and $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}}$ are arbitrary sequences of positive weights. Hence the reproducing kernels $K_s^{\boldsymbol{\eta}}$ and $L_s^{\boldsymbol{\gamma}}$ are given by

$$K_s^{\boldsymbol{\eta}}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{s} (1+\eta_j k(x_j,y_j)), \quad \mathbf{x},\mathbf{y} \in D^s,$$

and

$$L_s^{\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{s} (1+l_{\gamma_j}(x_j,y_j)), \quad \mathbf{x},\mathbf{y} \in D^s.$$

Here we present a particular consequence of Theorem 1 for summable weights, i.e.,

$$\sum_{j\in\mathbb{N}} \gamma_j < \infty. \tag{3}$$