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Approximation of multivariate periodic functions by trigonometric polynomials based on sampling along rank-1 lattice with generating vector of Korobov form

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ABSTRACT

In this paper, we present error estimates for the approximation of multivariate periodic functions in periodic Hilbert spaces of isotropic and dominating mixed smoothness by trigonometric polynomials. The approximation is based on sampling of the multivariate functions on rank-1 lattices. We use reconstructing rank-1 lattices with generating vectors of Korobov form for the sampling and generalize the technique from Temlyakov (1986), in order to show that the aliasing error of that approximation is of the same order as the error of the approximation using the partial sum of the Fourier series. The main advantage of our method is that the computation of the Fourier coefficients of such a trigonometric polynomial, which we use as approximant, is based mainly on a one-dimensional fast Fourier transform, cf. Kämmerer et al. (2013), Kämmerer (2014). This means that the arithmetic complexity of the computation depends only on the cardinality of the support of the trigonometric polynomial in the frequency domain. Numerical results are presented up to dimension $d = 10$.

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1. Introduction

We approximate functions $f \in \mathcal{H}^{\omega}(\mathbb{T}^d)$ from the Hilbert space

$$\mathcal{H}^{\omega}(\mathbb{T}^d) := \left\{ f \in L^1(\mathbb{T}^d) \colon \|f|\mathcal{H}^{\omega}(\mathbb{T}^d)\| := \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \omega(\boldsymbol{k})^2 |\hat{f}_{\boldsymbol{k}}|^2} < \infty \right\},$$

where $\omega \colon \mathbb{Z}^d \to (c, \infty]$, $c > 0$, is a weight function, by trigonometric polynomials p with frequencies supported on an index set $I \subset \mathbb{Z}^d$ of finite cardinality, $p(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i}\boldsymbol{k}\boldsymbol{x}}$. Thereby, we are especially interested in the higher-dimensional cases, i.e., $d \geq 4$. As usual, we denote the Fourier coefficients of the function f by

$$\hat{f}_{\boldsymbol{k}} := \int_{\mathbb{T}^d} f(\boldsymbol{x}) \mathrm{e}^{-2\pi \mathrm{i}\boldsymbol{k}\boldsymbol{x}} \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{k} \in \mathbb{Z}^d.$$

We remark that for the special choice $\omega \equiv 1$, we have $\mathcal{H}^{\omega}(\mathbb{T}^d) = L^2(\mathbb{T}^d)$. One theoretical possibility to obtain such a trigonometric polynomial p is to formally approximate the function f by the Fourier partial sum

$$S_I f := \sum_{\boldsymbol{k}\in I} \hat{f}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i}\boldsymbol{k}\circ},$$

where $I \subset \mathbb{Z}^d$ is a frequency index set of finite cardinality. Since $S_I f$ is the truncated Fourier series of the function f, this approximation causes a truncation error $\|f - S_I f\|$, where $\|\cdot\|$ is an arbitrarily chosen norm. For a function $f \in \mathcal{H}^{\omega}(\mathbb{T}^d)$ we choose a frequency index set $I = I_N := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \omega(\boldsymbol{k})^{1/\nu} \leq N\}$ of refinement $N \in \mathbb{R}$, $N \geq 1$, $\nu > 0$, and obtain

$$\|f - S_{I_N} f|L^2(\mathbb{T}^d)\| \leq N^{-\nu} \|f|\mathcal{H}^{\omega}(\mathbb{T}^d)\|,$$

see Lemma 3.3. We stress the fact that $S_{I_N} f$ is the best approximation of the function f with respect to the $L^2(\mathbb{T}^d)$ norm in the space $\Pi_{I_N} := \mathrm{span}\{\mathrm{e}^{2\pi \mathrm{i}\boldsymbol{k}\circ} \colon \boldsymbol{k} \in I_N\}$ of trigonometric polynomials with frequencies supported on the index set I_N and that the operator $S_{I_N} \colon L^1(\mathbb{T}^d) \to \Pi_{I_N}$ only depends on the frequency index set I_N. A similar estimate for the special case of product weights can be found in [18].

Since, in general, we do not know the Fourier coefficients $\hat{f}_{\boldsymbol{k}}$, we are going to approximate the function f from samples using the approximated Fourier partial sum

$$\tilde{S}_{I_N} f := \sum_{\boldsymbol{k}\in I_N} \tilde{\hat{f}}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i}\boldsymbol{k}\circ}.$$

We compute the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}} \in \mathbb{C}$, $\boldsymbol{k} \in I_N$, of the function f using sampling values. Therefore, we assume the function f to be continuous. We sample f along a rank-1 lattice and we compute the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$ by the rank-1 lattice rule

$$\tilde{\hat{f}}_{\boldsymbol{k}} := \frac{1}{M} \sum_{j=0}^{M-1} f\left(\boldsymbol{x}_j\right) \mathrm{e}^{-2\pi \mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} \quad \text{for } \boldsymbol{k} \in I_N, \tag{1.1}$$

where the sampling nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}$ are the nodes of a so-called reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$ with generating vector $\boldsymbol{z} \in \mathbb{Z}^d$ and rank-1 lattice size $M \in \mathbb{N}$ for the frequency index set I_N, see Section 2.2 for the definition. Lattice rules have extensively been investigated for the integration of functions of many variables for a long time, cf. e.g., [23,4,5] and the extensive reference list therein. Especially, rank-1 lattice rules have also been studied for the approximation of multivariate functions of suitable smoothness, cf. [26,19,18,20]. Furthermore, there exist already comprehensive tractability results for numerical integration and approximation using rank-1 lattices, see [21,18].

Since we consider the partial sum $\tilde{S}_{I_N} f$ of the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$ instead of the Fourier partial sum $S_{I_N} f$ of Fourier coefficients $\hat{f}_{\boldsymbol{k}}$, we obtain an additional error. As in [16], we