Journal of Complexity 31 (2015) 474–485

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

On the star discrepancy of sequences in the unit interval

Gerhard Larcher

Universität Linz, Institut für Finanzmathematik, Altenbergerstraße 69, Science Park Bauteil 2, A-4040 Linz, Germany

ARTICLE INFO

Article history:
Available online 7 August 2014

Keywords:
Star-discrepancy
Discrepancy constant
Uniform distribution

ABSTRACT

It is known that there is a constant $c > 0$ such that for every sequence $x_1, x_2, \ldots$ in $[0, 1)$ we have for the star discrepancy D_N^* of the first N elements of the sequence that $ND_N^* \geq c \cdot \log N$ holds for infinitely many N. Let c^* be the supremum of all such c with this property. We show $c^* > 0.0646363$, thereby improving the until now known estimates.

1. Introduction and statement of the result

Let $x_1, x_2, \ldots$ be a point sequence in $[0, 1)$. By D_N^* we denote the star discrepancy of the first N elements of the sequence, i.e.,

$$D_N^* = \sup_{x \in [0,1]} \left| \frac{\mathcal{A}_N(x)}{N} - x \right|, \quad \text{where}$$

$\mathcal{A}_N(x) := \#\{1 \leq n \leq N \mid x_n < x\}$.

The sequence $x_1, x_2, \ldots$ is uniformly distributed in $[0, 1)$ iff $\lim_{N\to\infty} D_N^* = 0$.

In 1972 Schmidt [5] has shown that there is a positive constant c such that for all sequences $x_1, x_2, \ldots$ in $[0, 1)$ we have

$$D_N^* > c \cdot \frac{\log N}{N}$$

for infinitely many N.

E-mail address: gerhard.larcher@jku.at.

http://dx.doi.org/10.1016/j.jco.2014.07.005
0885-064X/

The order $\frac{\log N}{N}$ in this result is best possible. There are many sequences known for which $D_N^* \leq c' \cdot \frac{\log N}{N}$ for a certain constant c' and for all N holds.

So it makes sense to define the "one-dimensional star discrepancy constant" c^* to be the supremum over all c such that

$$D_N^* > c \cdot \frac{\log N}{N}$$

holds for all sequences $x_1, x_2, \ldots$ in $[0, 1)$ for infinitely many N. Or, in other words

$$c^* := \inf_w \limsup_{N\to\infty} \frac{N D_N^*(w)}{\log N},$$

where the infimum is taken over all sequences $w = x_1, x_2, \ldots$ in $[0, 1)$, and $D_N^*(w)$ denotes the star discrepancy of the first N elements of w.

The currently best known estimates for c^* are

$$0.06015\ldots \leq c^* \leq 0.222\ldots.$$

The upper bound was given by Ostromoukhov [4] (thereby slightly improving earlier results of Faure (see for example [2])). The lower bound was given by Béjian [1]. (In fact Béjian derives his bound for c^* from a bound for the corresponding constant with respect to extreme discrepancy.)

It is the aim of this paper to give a simple, more illustrative proof of the result of Béjian on c^* with an even sharper lower bound for c^*.

We will prove

Theorem 1.1.

$$c^* \geq 0.0646363\ldots.$$

In Section 2 we will give some auxiliary results. The proof of Theorem 1.1 then follows in Section 3. The idea of the proof follows a method introduced by Liardet [3] which was also used by Tijdeman and Wagner in [6].

2. Auxiliary results

The first lemma was used in this context for the first time by Liardet in [3].

Lemma 2.1. *For any set A, any subsets A_0, A_2 of A and any function $f : A \to \mathbb{R}$ we have*

$$\max_{n\in A} f(n) - \min_{n\in A} f(n) \geq \frac{1}{2}\left(\max_{n\in A_2} f(n) - \min_{n\in A_2} f(n)\right) + \frac{1}{2}\left(\max_{n\in A_0} f(n) - \min_{n\in A_0} f(n)\right)$$
$$+ \frac{1}{2}\left|\max_{n\in A_2} f(n) - \max_{n\in A_0} f(n)\right| + \frac{1}{2}\left|\min_{n\in A_2} f(n) - \min_{n\in A_0} f(n)\right|.$$

Proof. This is quite elementary. □

Consider now a finite point set $x_1, x_2, \ldots, x_N$ in $[0, 1)$ with $N = [a^t]$, for some real a with $3 \leq a \leq 4$ and some $t \in \mathbb{N}$. Let A be the index-set $A = \{1, 2, \ldots, N\}$, and A_0, A_1, A_2 be the index-subsets

$$A_0 = \{1, 2, \ldots, [a^{t-1}]\}, \qquad A_2 = \{[a^t] - [a^{t-1}] + 1, [a^t] - [a^{t-1}] + 2, \ldots, [a^t]\} \quad \text{and}$$
$$A_1 = A \setminus (A_0 \cup A_2).$$

Assume first for simplicity that a^t and a^{t-1} are integers (of course this only can happen if $a = 3$ or $a = 4$).

For $x \in [0, 1)$ we consider the discrepancy function

$$D_n(x) := \#\{i \leq n \mid x_i < x\} - n \cdot x = \mathcal{A}_n(x) - n \cdot x.$$