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A remark on the numerical integration of harmonic functions in the plane

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ABSTRACT

We show that certain domains $\Omega \subset \mathbb{R}^2$ have the following property: there is a sequence of points $(x_i)_{i=1}^{\infty}$ in Ω with nonnegative weights $(a_i)_{i=1}^{\infty}$ such that for all harmonic functions $u : \mathbb{R}^2 \to \mathbb{R}$ and all $N \geq 1$ we have

$$\left| \int_{\Omega} u(x)dx - \sum_{i=1}^{N} a_i u(x_i) \right| \leq C_{\Omega} \frac{\|u\|_{L^{\infty}(\Omega)}}{N^{0.53}},$$

where C_{Ω} depends only on Ω. We emphasize that the points (x_i) and the weights (a_i) do not depend on u. This improves on the (probabilistic) Monte-Carlo bound $\|u\|_{L^2(\Omega)}/N^{0.5}$ *without* involving any sort of control on the oscillation of the function (which is classically done via the size of derivatives or the total variation). We do not know which decay rate is optimal.

1. Introduction

1.1. Harmonic functions

This paper aims to describe some progress in a problem that arose at the Oberwolfach Workshop 1340 'Uniform Distribution Theory and Applications', where it was mentioned by the author at the end of the talk describing [10]. We state the problem in its simplest possible setting: let $\Omega \subset \mathbb{R}^2$ be some bounded domain with a smooth boundary and let $u : \Omega \to \mathbb{R}$ be a *harmonic* function, i.e. assume it

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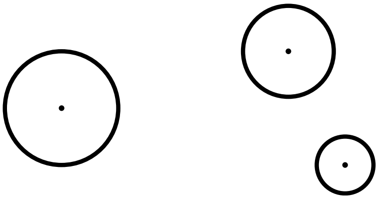

Fig. 1. A harmonic function can be exactly integrated in a disk with a single point evaluation. This means that for (the highly non-generic) domains that are given as the union of a finite number of disjoint disks, integration is simple.

satisfies

$$\Delta u = 0, \quad \text{where } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplacian. Is there any way of exploiting this information to effectively compute an approximation (including an error estimate) of

$$\int_{\Omega} u(x) dx?$$

Harmonic functions are infinitely many times differentiable: there is no problem with smoothness and all the classical methods of numerical integration are available: the real question is whether the additional information of the function being harmonic allows one to do better. The key ingredient suggesting that this might indeed be the case is the mean-value property: let $B(x, r)$ denote the disk with radius r centered at $x \in \mathbb{R}^2$. If u is harmonic in a neighborhood of $B(x, r)$, then

$$u(x) = \frac{1}{r^2\pi} \int_{B(x,r)} u(z) dz.$$

This means that exact integration over disks can be done with only one function evaluation (see Fig. 1)! However, the domains $\Omega \subset \mathbb{R}^2$ which can be decomposed into a finite number of disks are not very interesting and it is not immediately clear how this information could be suitably implemented on more general domains. We are not aware of any research precisely in that direction: there is, however, the vaguely related notion of a *quadrature domain* in complex analysis. A domain $\Omega \subset \mathbb{C}$ is called a quadrature domain, if there exist finitely many points $a_1, \dots, a_m \in \Omega$ and coefficients c_{kj} such that

$$\int_{\Omega} f(x) dx = \sum_{k=1}^{m} \sum_{j=0}^{n_k - 1} c_{kj} f^{(j)}(a_k)$$

for all analytic f. The example of the unit disk reduces to $m = 1$ and $n_1 = 1$ with a_1 being the center of the disk and c_{10} being its area. It is immediately obvious that these domains must be very special and indeed a domain is a quadrature domain if and only if the Cauchy transform

$$\hat{\chi}(z) = -\frac{1}{\pi} \int_{\Omega} \frac{d\zeta}{z - \zeta}$$

is a rational function outside of Ω (we refer to a survey of Gustafsson and Shapiro [4] for more information). This, however, does not seem to be applicable since we are interested in more general domains and do not wish to use any information on the gradient of the function. Another seemingly related field is that of optimal recovery problems (see, for example, the book of Osipenko [9]); these (often sharp) results, while related, also seem to depend too strongly on the underlying domain to be immediately applicable.