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Complexity of oscillatory integration for univariate Sobolev spaces

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ABSTRACT

We analyze univariate oscillatory integrals for the standard Sobolev spaces H^s of periodic and non-periodic functions with an arbitrary integer $s \geq 1$. We find matching lower and upper bounds on the minimal worst case error of algorithms that use n function or derivative values. We also find sharp bounds on the information complexity which is the minimal n for which the absolute or normalized error is at most ε. We show surprising relations between the information complexity and the oscillatory weight. We also briefly consider the case of $s = \infty$.

1. Introduction

We study the approximate computation of univariate oscillatory integrals (Fourier coefficients)

$$I_k(f) = \int_0^1 f(x)\, e^{-2\pi\, i\, kx}\, dx, \quad i = \sqrt{-1}, \tag{1}$$

where $k \in \mathbb{Z}$ and $f \in H^s$. We improve the known upper bounds and also prove matching lower bounds, i.e., we study the complexity of this computational problem. By H^s we mean the standard Sobolev (Hilbert) space; we study spaces of periodic and non-periodic functions defined on [0, 1] with an arbitrary integer $s \geq 1$. We usually consider a finite s but we also briefly consider the case

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of $s = \infty$. Although we consider arbitrary integers k, our emphasis is for large $|k|$ and we explain our results here only for such k.

We compute the initial error (the norm of I_k) as well as the worst case error of our algorithms exactly. This is possible since we assume that k is an integer. For the periodic case the initial error is of order $|k|^{-s}$, whereas for the non-periodic case it is independent of s and is roughly $|k|^{-1}$. This means that the initial error for the periodic case is much smaller for large s. For $s = \infty$, the periodic case leads to the space of only constant functions and the problem becomes trivial since the initial error is zero for all $k \neq 0$. The non-periodic case is still reasonable with the initial error roughly $|k|^{-1}$.

For a finite s and the periodic case, we prove that an algorithm that uses n function values at equally spaced points is nearly optimal, and its worst case error is bounded by $C_s(n + |k|)^{-s}$ with an exponentially small C_s in s. For the non-periodic case, we first compute successive derivatives up to order $s - 1$ at the end-points $x = 0$ and $x = 1$. These derivatives values are used to periodize the function and this allows us to obtain similar error bounds like for the periodic case. Asymptotically in n, the worst case error of the algorithm is of order n^{-s} independent of k for both periodic and non-periodic cases.

Near optimality of this algorithm is shown by proving a lower bound of order $(n + |k|)^{-s}$ which holds for all algorithms that use the values of function and derivatives up to order $s-1$ at n arbitrarily chosen points from $[0, 1]$. We establish the lower bound by constructing a periodic function that vanishes with all its derivatives up to order $s - 1$ at the points sampled by a given algorithm, belongs to the unit ball of the space H^s, and its oscillatory integral is of order $(n + |k|)^{-s}$.

For $s = \infty$, we provide two algorithms which compute successive derivatives and/or function values at equally spaced points. The worst case error of one of these algorithms is super exponentially small in n. For $s = \infty$, we do not have a matching lower bound.

We consider the absolute and normalized error criteria. For the absolute error criterion, we want to find the information complexity which is defined as the smallest n for which the nth minimal error is at most $\varepsilon \in (0, 1)$, whereas for the normalized error criterion, the information complexity is the smallest n for which the nth minimal error reduces the initial error by a factor ε. For a finite s we obtain the following results.

- For the absolute error criterion and the periodic case, the information complexity is zero if $\varepsilon > 1/(2\pi |k|)^s$ and otherwise is roughly $\varepsilon^{-1/s} - |k|$. This means that in this case the problem becomes easier for large $|k|$.
- For the normalized error criterion and for the periodic case, the information complexity is of order $|k|\,\varepsilon^{-1/s}$. Hence, in this case the problem becomes harder for larger $|k|$.
- For the absolute error criterion and the non-periodic case, the information complexity is zero if $\varepsilon \geq 1.026/(2\pi |k|)$ and otherwise is roughly lower bounded by $\varepsilon^{-1/s} - |k|$ and upper bounded by $\varepsilon^{-1/s} + 2s - 1 - |k|$. As for the periodic case, the problem becomes easier for large k.
- For the normalized error criterion and the non-periodic case, the information complexity is of order $|k|^{1/s}\,\varepsilon^{-1/s}$ for very small ε. In this case, the dependence on $|k|$ is more lenient than for the periodic case especially if s is large.

  The dependence on $|k|$ is quite intriguing if $|k|$ goes to infinity. For $s = 1$ and fixed ε, the information complexity goes to infinity linearly with $|k|$. However, the situation is quite different for $s \geq 2$. Then for large $|k|$ the information complexity is bounded by $2s$ if ε is fixed or if ε tends to zero like $|k|^{-\eta}$ with $\eta \in (0, s - 1)$.

For $s = \infty$, we obtain only upper bounds on the information complexity. For ε tending to zero they are roughly $\ln(\varepsilon^{-1})/\ln(\ln(\varepsilon^{-1}))$ independent of $|k|$.

There are several recent papers about the approximate computation of highly oscillatory univariate integrals with the weight $\exp(2\pi\, i\, kx)$, where $x \in [0, 1]$ and k is an integer (or $k \in \mathbb{R}$) which is assumed to be large in the absolute sense, see Domínguez, Graham and Smyshlyaev [4], Iserles and Nørsett [6], Melenk [8], Chapter 3 of Olver [11], and Huybrechs and Olver [5] for a survey. Some authors mainly present asymptotic error bounds as k goes to infinity for algorithms that use n function or derivative values. It is usually done for C^∞ or even analytic functions. There are not too many papers that contain explicit error bounds depending on k and n. Examples include [4,8,11]. All these papers also contain pointers to the further relevant literature.