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The descriptive complexity of stochastic integration

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ABSTRACT

For most functionals for which pathwise stochastic integration with respect to Brownian motion is defined, sample Brownian paths for which the integral exists are very hard to construct. There exist on the unit interval, functions ω that can be uniformly approximated by sequences of continuous piece-linear functions (ω_n) such that each ω_n is encoded by a finite binary string of high Kolmogorov–Chaitin complexity. Such functions ω are called complex oscillations. Their set has Wiener measure 1 and they are fully characterised by infinite binary strings of high complexity. In this paper we study stochastic integration from the point of view of complex oscillations. We prove that, under some computability properties on integrands, pathwise stochastic integrals exist for *any* complex oscillation. We prove also that Itô's lemma holds for *each* complex oscillation. Thus constructing a continuous function satisfying Itô's lemma is reduced to constructing an infinite binary string of high Kolmogorov–Chaitin complexity.

1. Introduction

A Brownian motion on the unit interval is a real-valued function $(t, \omega) \mapsto X(t, \omega)$ on $[0, 1] \times \Omega$, where Ω is the underlying space of some probability space, such that $X(0, \omega) = 0$, the function $t \mapsto X(t, \omega)$ is continuous for any ω, and for any finite sequence $0 < t_1 < \cdots < t_n$ in the unit interval, the random variables $\omega \mapsto X(t_1, \omega), X(t_2, \omega) - X(t_1, \omega), \ldots, X(t_n, \omega) - X(t_{n-1}, \omega)$ are statistically independent and normally distributed with means 0 and variances $t_1, t_2 - t_1, \ldots, t_n - t_{n-1}$, respectively. The canonical Brownian motion X, that will be assumed throughout, is obtained by

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taking $\Omega = C[0,1]$, the set of continuous real functions defined on $[0,1]$ and vanishing at the origin, with its uniform norm topology and endowed with the *Wiener measure* P, and X is defined by $X(t,\omega) = \omega(t)$. The random variable $\omega \mapsto X(t,\omega)$ will be denoted $X(t)$. We recall that P is the unique Borel probability measure on $C[0,1]$ such that for any $0 < t_1 < \cdots < t_n \le 1$ and any Borel subset A of $\mathbf{R}^n$,

$$P\left[\omega \in C[0,1] : (\omega(t_1), \ldots, \omega(t_n)) \in A\right] = \int_A \frac{e^{-\frac{1}{2}x^T Q^{-1} x}}{(2\pi)^{n/2}(\det Q)^{1/2}} dx$$

where $Q = (q_{ij})$ is the $n \times n$ matrix defined by $q_{ij} = t_i$, for $i \le j$. (More discussions can be found in [13].)

As it is well known, most probability theory results are almost surely properties in the sense that they hold outside exceptional sets of null probability. Generally, it is difficult to say more than that the probability of an event is 1. Very little is known about these exceptional null sets and it seems that probability theory gives them less interest. On the other hand, one can ask if probability theory (or measure theory in general) is well equipped to analyse such specific objects. Typical Brownian motion properties (or stochastic processes in general) are surprisingly unexpected, counter-intuitive and in general with non-constructive proofs. A natural question that arises when studying a typical property of Brownian motion is to know how difficult is it to construct a "sample path" satisfying that property.

Generally, sets of Wiener measure 1 are defined by a given particular property of Brownian motion. Asarin and Prokovskii [1] were the first to define a subset of $C[0,1]$ of full Wiener measure independently of any specific Brownian motion property by using the notion of Kolmogorov–Chaitin complexity. They considered the set of functions on $[0,1]$ that can be uniformly approximated by sequences of piece-linear functions encoded by finite binary strings of high Kolmogorov–Chaitin complexity (precise definitions are given in Section 2). Such functions are now referred to as algorithmically random Brownian motions or complex oscillations. Fouché [7] has completely characterised complex oscillations: any complex oscillation is uniquely and recursively determined by an infinite binary string α such that for a fixed constant d, the Kolmogorov–Chaitin complexity of the substring $\bar{\alpha}(n)$ of the first n bits of α is at least $n - d$ for all n. (The set of such infinite binary strings will be denoted KC.)

The major contribution of Asarin and Prokovskii is that they provided a playground where questions related to complexity of stochastic processes can be analysed. The concept of Kolmogorov–Chaitin complexity has already played a major role in defining randomness properties in the classical Cantor space of binary strings. The interplay between algorithmic randomness and probability theory is very useful in obtaining effective probability properties. (See for example Davie [5] and Hoyrup and Rojas [12].)

Fouché [8] was the first to observe that the set (of complex oscillations) introduced by Asarin and Prokovskii can be used to obtain effective Brownian motion laws in the same way complex binary strings are used to obtain effective probability laws in the classical Cantor set $\{0,1\}^{\mathbf{N}}$. He proposed to study Brownian motion properties from the point of view of complex oscillations. Fouché [7] proved that *every* complex oscillation satisfies the well-known modulus continuity property of Brownian motion.

In this paper we continue the study of effective Brownian motion properties by considering the key notion of stochastic integration. Stochastic calculus (or Itô calculus) is an important technique of probability theory with many applications in financial mathematics, telecommunications, etc.

Given a functional $f : [0,1] \times C[0,1] \to \mathbf{R}$ depending on $t \in [0,1]$ and Brownian motion paths $\omega \in C[0,1]$, we shall consider the sums

$$S_n(f,t,\omega) = \sum_{k=1}^{l} f((k-1)2^{-n},\omega)(\omega(k2^{-n}) - \omega((k-1)2^{-n})) + f(l2^{-n},\omega)(\omega(t) - \omega(l2^{-n}))$$

where $l = [2^n t]$, the largest integer $\le 2^n t$. If the limit $\lim_{n\to\infty} S_n(f,t,\omega)$ exists for $t \in [0,1]$, we will say that the limit is the (pathwise) stochastic integral of f with respect to the path $\omega \in C[0,1]$ on $[0,t]$, denoted $\int_0^t f(s,\omega)d\omega(s)$. If the limit exists almost surely for all ω and simultaneously for all $t \in [0,1]$, then it defines a version of stochastic integral on $[0,1]$. The stochastic integral of a given