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A condition-based algorithm for solving polyhedral feasibility problems

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ABSTRACT

It is known that *Polyhedral Feasibility Problems* can be solved via interior-point methods whose real number complexity is polynomial in the dimension of the problem and the logarithm of a *condition number* of the problem instance. A limitation of these results is that they do not apply to *ill-posed* instances, for which the condition number is infinite. We propose an algorithm for solving polyhedral feasibility problems in homogeneous form that is applicable to all problem instances, and whose real number complexity is polynomial in the dimension of the problem instance and in the logarithm of an "*extended condition number*" that is always finite.

1. Introduction

We propose an algorithm to solve the polyhedral feasibility problems

$$Ax = 0, \quad x \geq 0, \; x \neq 0, \tag{1}$$

and

$$A^{\mathrm{T}}y \geq 0, \qquad A^{\mathrm{T}}y \neq 0, \tag{2}$$

where A is a matrix in $\mathbb{R}^{m\times n}$. We refer to problems (1) and (2) as the *feasibility pair* defined by matrix A. A given matrix A is well-posed if (1) or (2) is strictly feasible and remains so under sufficiently small

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perturbations of the inputs. A matrix A is ill-posed when arbitrarily small perturbations on A can yield a change in the feasibility status of the problems (1) and (2).

When the matrix A is well-posed, a strict feasible solution of (1) or (2) can be found by solving linear programming reformulations of these systems. The difficulty of solving (1) and (2) depends on how well-posed the matrix A is. Renegar [11], Peña and Renegar [10], Freund and Vera [6], Epelman and Freund [5], and Bürgisser and Cucker [1] have developed algorithms whose complexity depends on a condition number of the instance A that is a measure of the distance to ill-posedness. In particular, Bürgisser and Cucker [1] consider an interior-point method that has the best known condition-based convergence rate and finds a strictly feasible solution to (1) or (2) in $\mathcal{O}\left(\sqrt{n}\,[\log n + \log \mathcal{C}(A)]\right)$ iterations, where $\mathcal{C}(A)$ is Goffin–Cucker–Cheung's condition number [2,7]. The condition number $\mathcal{C}(A)$ is finite for well-posed instances but is infinite for ill-posed matrices.

When the matrix A is ill-posed and full row-rank, neither problem in the feasibility pair (1) and (2) has strict solutions. Instead, there exists a unique partition $\Omega(A) = (B, N)$ of $B \cup N = \{1, \ldots, n\}$ such that both systems

$$A_B x_B = 0, \quad x_B > 0, \qquad x_N = 0, \tag{3}$$

and

$$A_N^{\mathrm{T}} y > 0, \qquad A_B^{\mathrm{T}} y = 0, \tag{4}$$

are feasible. Here A_B denotes the sub-matrix of A that only contains the columns of A whose indices are in B. The sub-matrix A_N, and sub-vectors x_B and x_N are similarly defined. The partition $\Omega(A)$ is called the Goldman and Tucker partition and its existence can be proved using the Farkas' Lemma [3]. When the matrix A is well-posed the partition $\Omega(A)$ is trivial, that is, $N = \emptyset$ or $B = \emptyset$. We say A is *primal feasible* if $N = \emptyset$ and it is *dual feasible* if $B = \emptyset$.

We propose an interior-point algorithm that finds the partition $\Omega(A)$ and the corresponding solutions $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ that satisfy (3) and (4), respectively. We show that our algorithm halts in at most $\mathcal{O}\left(\sqrt{n}[\log n + \log \bar{\mathcal{C}}(A)]\right)$ iterations where the condition number $\bar{\mathcal{C}}(A)$ proposed by Cheung et al. [4] is an extension of $\mathcal{C}(A)$. Unlike $\mathcal{C}(A)$, the condition number $\bar{\mathcal{C}}(A)$ is finite for ill-posed instances. Our complexity result establishes that Bürgisser and Cucker's algorithmic complexity [1] for well-posed matrices can also be achieved in the ill-posed setting with a modified condition number.

Vavasis and Ye [14] have also proposed an algorithm that finds a solution to (1) and (2) in $\mathcal{O}(\sqrt{n}[\log n + \log \chi(A)])$ for certain parameter $\chi(A)$ that is finite for all matrices A. However, the parameter $\chi(A)$ is not defined in terms of the sensitivity of the problem to data perturbations on A. By contrast, as its predecessor $\mathcal{C}(A)$, the condition number $\bar{\mathcal{C}}(A)$ used in our analysis can be defined in terms of how perturbations in A generate a change in the behavior of solutions to the problems (1) and (2) (See Section 2 for more details). Therefore, our approach is better aligned for perturbation theory and round-off analysis.

This paper is organized as follows: We begin by recalling the condition numbers $\mathcal{C}(A)$ and $\bar{\mathcal{C}}(A)$. We recast the pair (1)–(2) as a primal–dual pair of linear optimization programs in Section 3. Our main result is given in Section 4, where we describe our interior-point algorithm and its iteration bound complexity. Proofs associated with our algorithm are given in Section 5.

2. Condition numbers

Assume $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{m \times n}$ with $a_i \neq 0$ for all $i = 1, \ldots, n$ and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^m$. Let

$$\rho(A) := \left| \max_{y \neq 0} \min_{j=1,\ldots,n} \frac{a_j^{\mathrm{T}} y}{\|a_j\| \, \|y\|} \right|. \tag{5}$$

The parameter $\rho(A)$ is motivated by the most interior solution of the system $A^{\mathrm{T}} y > 0$, when this problem is feasible. More precisely, if $A^{\mathrm{T}} y > 0$ is feasible then $\rho(A)$ is the *width* of the feasibility cone $\{y : A^{\mathrm{T}} y > 0\}$ as defined by Freund and Vera [6]. Furthermore, $\rho(A)$ is the Euclidean distance