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Uniform weak tractability of multivariate problems with increasing smoothness

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ABSTRACT

We study d-variate approximation problems with varying regularity with respect to successive variables. The varying regularity is described by a sequence of real numbers $\{r_k\}_{k\in\mathbb{N}}$ satisfying

$$0 \le r_1 \le r_2 \le r_3 \le \cdots.$$

We mainly consider algorithms that use finitely many continuous linear functionals. In the worst case setting we study approximation problems defined over suitable Korobov and Sobolev spaces. In the average case setting we study approximation problems defined over the space of continuous functions $C([0,1]^d)$ equipped with a zero-mean Gaussian measure whose covariance operator is given by an Euler or Wiener integrated process. We establish necessary and sufficient conditions on uniform weak tractability of those problems in terms of their regularity parameters $\{r_k\}_{k\in\mathbb{N}}$.

1. Introduction

Tractability of multivariate problems studies the intrinsic difficulty of problems defined on spaces of d-variate functions. By a problem we understand a sequence $S = \{S_d\}_{d\in\mathbb{N}}$ of operators, such that for every d the operator S_d acts on a suitable space of d-variate functions. The intrinsic difficulty of a problem S is measured by its information complexity, $n(\varepsilon, S_d)$, which is defined as the minimal number of information operations needed to obtain an ε-approximation of the solution of the d-th

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instance of the problem S. By one information operation we mainly mean one continuous linear functional. We also briefly mention the case when one information operation is given by one function value. If the function $n(\varepsilon, S_d)$ depends exponentially on ε^{-1} or d we say that the problem S is intractable. The tractable problems, that is those problems S with the information complexity $n(\varepsilon, S_d)$ not exponential in ε^{-1} and/or d, are subject of further classification. Depending on the behavior of their information complexity with respect to ε and d, problems occupy an adequate place in the tractability hierarchy of multivariate problems. As in [7], we say that the problem S is:

- *strongly polynomially tractable* (SPT) iff there are non-negative numbers C and p such that

$$n(\varepsilon, S_d) \le C\,\varepsilon^{-p} \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

  The infimum of such p is called the exponent of SPT and denoted by p^*.
- *polynomially tractable* (PT) iff there are non-negative numbers C, p and q such that

$$n(\varepsilon, S_d) \le C\,\varepsilon^{-p}\, d^{\,q} \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

As in [2], we say that S is

- *quasi-polynomially tractable* (QPT) iff there are non-negative numbers C and t such that

$$n(\varepsilon, S_d) \le C\,\exp\big(t\,(1+\ln\,\varepsilon^{-1})(1+\ln\,d)\big) \quad \text{for all } \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

  The infimum of such t is called the exponent of QPT and denoted by t^*.

As in [12], we say that S is

- *uniformly weakly tractable* (UWT) iff

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln\, n(\varepsilon, S_d)}{\varepsilon^{-\alpha} + d^{\beta}} = 0 \quad \text{for all } \alpha, \beta > 0.$$

  We add in passing that it is enough to check the last condition for all $\alpha = \beta > 0$.

As in [7], we say that S is

- *weakly tractable* (WT) iff the last condition holds for $\alpha = \beta = 1$.

Clearly,

$$\text{SPT} \Longrightarrow \text{PT} \Longrightarrow \text{QPT} \Longrightarrow \text{UWT} \Longrightarrow \text{WT}.$$

More on tractability including the motivation of tractability studies can be found in [7–9].

Multivariate problems for which all the variables are equally important are often intractable. In particular, many multivariate problems suffer from the curse of dimensionality, i.e., their information complexity is an exponential function of the number d of variables. One of the ways of vanquishing the curse of dimensionality is the introduction of non-homogeneity to the structure of the problem. The non-homogeneity may be introduced to a problem by means of weights associated with the importance of variables and groups of variables, or by means of varying regularity of a problem with respect to successive variables. Those approaches have been recently subject to an intense research.

In this paper we further investigate the relationship between tractability of a problem and its increasing regularity with respect to successive variables. We study problems with unknown UWT, and sometimes with unknown QPT. The relationship between the other notions of tractability and increasing regularity of a problem have already been studied in [11] in the worst case setting, and in [5,6] in the average case setting.

We deal with the problem of approximation of functions with increasing regularity with respect to successive variables. In the worst case setting the problem is the approximation of functions from suitable Korobov spaces or Sobolev spaces. In the average case setting the problem is the approximation of continuous functions equipped with a zero-mean Gaussian measure with covariance operator given by integrated Euler process or integrated Wiener process. Those zero-mean Gaussian measures are concentrated on spaces of functions with suitably increasing regularity with respect to successive variables.