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Tractability of linear problems defined over Hilbert spaces[☆]

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ABSTRACT

We study d-variate approximation problems in the worst and average case settings. We consider algorithms that use finitely many evaluations of arbitrary linear functionals. In the worst case setting, we obtain necessary and sufficient conditions for quasi-polynomial tractability and uniform weak tractability. Furthermore, we give an estimate of the exponent of quasi-polynomial tractability which cannot be improved in general. In the average case setting, we obtain necessary and sufficient conditions for uniform weak tractability. As applications we discuss some examples.

1. Introduction

Multivariate computational problems are defined on classes of functions depending on d variables with large or even huge d. Multivariate problems occur in many applications such as in computational finance, statistics and physics. Such problems are usually solved by algorithms that use finitely many information operations. One information operation is defined as one function value or the evaluation of one linear functional. The minimal number of information operations needed to find the solution to within ε, independently of the information operations and algorithms, is the intrinsic difficulty of the problem. It is called the information complexity and is denoted by $n(\varepsilon, d)$ to stress its dependence on the two important parameters.

Research on tractability of multivariate continuous problems started in 1994 (see [13]). The purpose of tractability is to study the complexity with respect to ε^{-1} and d. Tractability of multivariate problems has been studied for different error criteria and in different settings including the worst and

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average case settings. Different kinds of tractable problems have been considered in the literature. In fact, tractability of multivariate problems has been recently a very active research area: see [5–7] and the references therein. A problem is intractable if the information complexity is an exponential function of ε^{-1} or d. Otherwise, the problem is tractable. We list the different tractability concepts below:

- weak tractability if $n(\varepsilon, d)$ is not exponential in d or ε^{-1}.
- uniform weak tractability if $n(\varepsilon, d)$ is not exponential in any power of d and ε^{-1}.
- quasi-polynomial tractability if $n(\varepsilon, d)$ is of order $\exp(t(1 + \ln d)(1 + \ln \varepsilon^{-1}))$ for some $t > 0$.
- polynomial tractability if $n(\varepsilon, d)$ is of order $d^q \varepsilon^{-p}$ for some $p, q > 0$.
- strong polynomial tractability if $n(\varepsilon, d)$ is of order ε^{-p} for some $p > 0$.

The bounds above hold for all $d \in \mathbb{N}$ and all $\varepsilon \in (0, 1)$ with the parameters t, q, p and the prefactors independent of d and ε.

The concept of quasi-polynomial tractability was introduced recently in [1]. Similar investigations can be found in [2–4,14]. One of the main goals of this paper is to study quasi-polynomial tractability of linear problems in the worst case setting, and this is done for information consisting of arbitrary linear functionals.

The concept of uniform weak tractability was also introduced recently in [11], which mainly discusses linear tensor product problems. Another goal of this paper is to study uniform weak tractability of linear problems defined over Hilbert spaces. This is also done for information consisting of arbitrary linear functionals.

In Section 2, we introduce some basic concepts and results.

In Section 3, we study quasi-polynomial tractability of general linear multivariate problems defined over Hilbert spaces in the worst case setting. We consider the problem $S = \{S_d\}$, where $S_d : H_d \to G_d$ is a compact linear operator and H_d and G_d are Hilbert spaces. We find necessary and sufficient conditions in terms of the eigenvalues of $W_d = S_d^* S_d : H_d \to H_d$ (S_d^* denotes the adjoint operator of S_d) for both the absolute error criterion and the normalized error criterion. Besides, we obtain an estimate of the exponent t^{qpol} of quasi-polynomial tractability which cannot be improved in general.

In Section 4, we study uniform weak tractability of general linear multivariate problems defined over Hilbert spaces in the worst case setting. We find necessary and sufficient conditions in terms of the eigenvalues of W_d for both the absolute error criterion and the normalized error criterion.

In Section 5, we study uniform weak tractability of general linear multivariate problems defined over Hilbert spaces in the average case setting. In this case, it is well known that the optimal algorithm and its error can be expressed by the eigenvectors and eigenvalues of the covariance operator of the measure, respectively. Hence, we find necessary and sufficient conditions in terms of the eigenvalues of the covariance operator, for both the absolute error criterion and the normalized error criterion.

2. Some concepts

We will use terminology from [5–7]. Assume we are given a sequence of solution operators

$$S_d : F_d \to G_d \quad \text{for all } d \in \mathbb{N}.$$

Here, F_d is a subset of some normed space H_d, and G_d is a normed space. We approximate $S_d f, f \in F_d$ by algorithms

$$A_{n,d}(f) = \phi_{n,d}(L_1(f), \dots, L_n(f)),$$

where $L_j \in H_d^*$ (here and in the following H_d^* denotes the dual space of normed space H_d) and $\phi_{n,d} : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary mapping. The error of the algorithm $A_{n,d}$ is defined as

$$e(A_{n,d}) = \sup_{f \in F_d} |S_d(f) - A_{n,d}(f)|.$$

For $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$, let $n(\varepsilon, d)$ be the information complexity which is defined as the minimal number of continuous linear functionals which are necessary to obtain an ε-approximation of S_d in the worst case for the absolute or normalized error criterion (see [5, p. 106]), i.e.,

$$n(\varepsilon, d) = \min\{n \in \mathbb{N} | \exists A_{n,d} \text{ such that } e(A_{n,d}) \le \varepsilon \mathrm{CRI}_d\},$$