Journal of Complexity 30 (2014) 750–766

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

Complexity of parametric integration in various smoothness classes

Thomas Daun, Stefan Heinrich *

Department of Computer Science, University of Kaiserslautern, D-67653 Kaiserslautern, Germany

ARTICLE INFO

Article history:
Received 22 August 2013
Accepted 15 April 2014
Available online 24 April 2014

Keywords:
Parametric integration
Multilevel Monte Carlo method
Dominating mixed smoothness
Lower bounds
Complexity

ABSTRACT

We continue the complexity analysis of parametric definite and indefinite integration given by Daun and Heinrich (2013). Here we consider anisotropic classes of functions, including certain classes with dominating mixed derivatives. Our analysis is based on a multilevel Monte Carlo method developed by Daun and Heinrich (2013) and we obtain the order of the deterministic and randomized n-th minimal errors (in some limit cases up to logarithms). Furthermore, we compare the rates in the deterministic and randomized setting to assess the gain reached by randomization.

1. Introduction

The complexity of definite parametric integration was studied in [10,6,16], while in [2] the complexity of both definite and indefinite parametric integration was considered. Parametric definite integration is a problem intermediate between integration and approximation. Parametric indefinite integration can be viewed as a model for the solution of parametric initial value problems in the sense that it is a partial, but typical case, and some of the methods developed here will be used in the study of parametric initial value problems, see [3].

This paper is a continuation of [2] and we study both definite and indefinite integration. So far definite parametric integration was considered only for isotropic classes and, in [6], for a specific anisotropic class (Sobolev case with no smoothness in the integration variable). Indefinite parametric integration was only studied for C^r. In [2] we gave a general (multilevel) scheme for Banach space valued integration of functions belonging to

$$C^r(X) \cap C^{r_1}(Y), \tag{1}$$

* Corresponding author.
E-mail addresses: daun@informatik.uni-kl.de (T. Daun), heinrich@informatik.uni-kl.de (S. Heinrich).

http://dx.doi.org/10.1016/j.jco.2014.04.002
0885-064X/

where X and Y are Banach spaces such that Y is continuously embedded into X, from which the upper bounds for parametric integration in the C^r-case were derived.

In the present paper we further explore the range given in (1) by considering classes of functions with dominating mixed derivatives and other types of non-isotropic smoothness. In contrast to the C^r case, these classes allow to treat different smoothnesses for the parameter dependence and for the basic (nonparametric) integration problem. We want to understand the typical behavior of the complexity in these classes and the relation between the deterministic and randomized setting, this way clarifying in which cases and to which extend randomized methods are superior to deterministic ones.

The paper is organized as follows. In Section 3 we recall the needed algorithms and results for Banach space valued definite and indefinite integration from [2]. In Section 4 we consider parametric definite and indefinite integration and obtain the main results. Applications to various smoothness classes are given in Section 5, together with some comments on the relation between the deterministic and the randomized setting.

2. Preliminaries

We denote $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Given Banach spaces X, Y, we let $\mathscr{L}(X, Y)$ be the space of bounded linear operators from X to Y, equipped with the usual norm, and we write $\mathscr{L}(X)$ if $X = Y$. The dual space of X is denoted by X^*, the identity mapping on X by I_X, and the closed unit ball by B_X. The norm of X is denoted by $\|\cdot\|$, other norms are distinguished by subscripts. We assume all considered Banach spaces to be defined over the same scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

We often use the same symbol for possibly different constants. Given two sequences of nonnegative reals $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, the notation $a_n \preceq b_n$ means that there are constants $c > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $a_n \le c b_n$. Moreover, we write $a_n \asymp b_n$ if $a_n \preceq b_n$ and $b_n \preceq a_n$. We also use the notation $a_n \asymp_{\log} b_n$ if there are constants $c_1, c_2 > 0$, $n_0 \in \mathbb{N}$, and $\theta_1, \theta_2 \in \mathbb{R}$ with $\theta_1 \le \theta_2$ such that for all $n \ge n_0$

$$c_1 b_n (\log(n+1))^{\theta_1} \le a_n \le c_2 b_n (\log(n+1))^{\theta_2}.$$

Throughout the paper log means $\log_2$.

For $1 \le p \le 2$ a Banach space X is said to be of (Rademacher) type p, if there is a constant $c \ge 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$

$$\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|^p \le c^p \sum_{k=1}^n \|x_i\|^p, \tag{2}$$

with $(\varepsilon_i)_{i=1}^n$ being independent random variables satisfying $\mathbb{P}\{\varepsilon_i = -1\} = \mathbb{P}\{\varepsilon_i = +1\} = 1/2$. The type p constant $\tau_p(X)$ of X is the smallest constant $c \ge 0$ satisfying (2), and $\tau_p(X) = \infty$, if there is no such c. We refer to [11] for background on this notion. The space $L_{p_1}(\mathcal{M}, \mu)$, where $(\mathcal{M}, \mu)$ is an arbitrary measure space and $p_1 < \infty$, is of type p with $p = \min(p_1, 2)$. Furthermore, there is a constant $c > 0$ such that $\tau_2(\ell_\infty^n) \le c(\log(n+1))^{1/2}$ for all $n \in \mathbb{N}$.

Let $Q = [0,1]^d$ and let $C^r(Q, X)$ denote the space of all r-times continuously differentiable functions $f : Q \to X$ equipped with the norm

$$\|f\|_{C^r(Q,X)} = \max_{\alpha\in\mathbb{N}_0^d,\, |\alpha|\le r,\, t\in Q} \left\|\frac{\partial^{|\alpha|} f(t)}{\partial t^\alpha}\right\|.$$

For $r = 0$ we write $C^0(Q, X) = C(Q, X)$, which is the space of continuous X-valued functions on Q, and if $X = \mathbb{K}$, we write $C^r(Q)$ and $C(Q)$.

Let $X \otimes Y$ be the algebraic tensor product of Banach spaces X and Y and let $X \otimes_\lambda Y$ be the injective tensor product, defined as the completion of $X \otimes Y$ with respect to the norm

$$\lambda\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sup_{u\in B_{X^*},\, v\in B_{Y^*}} \left|\sum_{i=1}^n \langle x_i, u\rangle \langle y_i, v\rangle\right|.$$

Background on tensor products can be found in [4,12]. For Banach spaces X_1, Y_1 and operators $T \in \mathscr{L}(X, X_1)$, $U \in \mathscr{L}(Y, Y_1)$, the algebraic tensor product $T \otimes U : X \otimes Y \to X_1 \otimes Y_1$ extends to a bounded