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A semilocal convergence result for Newton's method under generalized conditions of Kantorovich

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ABSTRACT

From Kantorovich's theory we establish a general semilocal convergence result for Newton's method based fundamentally on a generalization required to the second derivative of the operator involved. As a consequence, we obtain a modification of the domain of starting points for Newton's method and improve the a priori error estimates. Finally, we illustrate our study with an application to a special case of conservative problems.

1. Introduction

It is well-known that solving equations of the form $F(x) = 0$ where F is a nonlinear operator, $F : \Omega \subseteq X \to Y$, defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y, is a very common problem in engineering and science. Although some equations can be solved analytically, we usually look for numerical approximations of the solutions, since finding exact solutions is usually difficult. To approximate a solution of $F(x) = 0$ we normally use iterative methods and Newton's method,

$$\begin{cases} x_0 \in \Omega, \\ x_n = x_{n-1} - [F'(x_{n-1})]^{-1}F(x_{n-1}), \quad n \in \mathbb{N}, \end{cases} \tag{1}$$

is one of the most used because of its simplicity, easy implementation and efficiency.

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The first semilocal convergence result for Newton's method in Banach spaces is due to L.V. Kantorovich, which is usually known as the Newton–Kantorovich theorem [12] and is proved under the following conditions for the operator F and the starting point x_0:

(C_1) There exists $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ for some $x_0 \in \Omega$, $\|\Gamma_0\| \leq \beta$ and $\|\Gamma_0 F(x_0)\| \leq \eta$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y to X,
(C_2) $\|F''(x)\| \leq \ell$, for $x \in \Omega$,
(C_3) $\ell\beta\eta \leq \frac{1}{2}$ and $B\left(x_0, \frac{1-\sqrt{1-2\ell\beta\eta}}{\ell\beta}\right) \subset \Omega$.

Since then, many papers have appeared that study the semilocal convergence of the method. Most of them are modifications of the Newton–Kantorovich theorem in order to relax conditions (C_1)–(C_3), specially condition (C_2). In [6], we present a first generalization of condition (C_2), where we replace (C_2) by the condition

$$\|F''(x)\| \leq \omega(\|x\|), \quad x \in \Omega, \tag{2}$$

where $\omega : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ is a non-decreasing continuous real function. Obviously, condition (2) generalizes condition (C_2). From the function ω, we construct a scalar function that allows us to define a majorizing sequence of Newton's method in Banach spaces, so that the convergence of Newton's method in Banach spaces is guaranteed from the majorizing sequence. The main aim of this paper is to generalize condition (2) to successive derivatives of the operator F.

The paper begins in Section 2 by recalling the concept of majorizing sequence and presenting the Newton–Kantorovich theorem. In this section we also introduce the new convergence conditions for the general case. In Section 3, we present a new general semilocal convergence theorem for Newton's method and indicate how the majorizing sequences are constructed. We also include information about the existence and uniqueness of solution and a result on the a priori error estimates that leads to the quadratic convergence of Newton's method. In Section 4, we give a particular case of our general result. Finally, in Section 5, we present an application where a conservative problem is involved. We clearly show the advantages of our new semilocal convergence result with respect to the Newton–Kantorovich theorem.

Throughout the paper we denote $\overline{B(x, \varrho)} = \{y \in X; \|y - x\| \leq \varrho\}$ and $B(x, \varrho) = \{y \in X; \|y - x\| < \varrho\}$.

2. Preliminary information

The known Newton–Kantorovich theorem [12] guarantees the semilocal convergence of Newton's method in Banach spaces and gives a priori error estimates and information about the existence and uniqueness of the solution. Kantorovich proves the theorem by using two different techniques [10,11], although the most prominent one is the majorant principle [11], which is based on the concept of majorizing sequence. This technique has been usually used later by other authors to analyze the semilocal convergence of several iterative methods [2,1,4,16]. We begin by introducing the concept of majorizing sequence and remembering how it is used to prove the convergence of sequences in Banach spaces.

Definition 1. If $\{x_n\}$ is a sequence in a Banach space X and $\{t_n\}$ is a scalar sequence, then $\{t_n\}$ is a *majorizing sequence* of $\{x_n\}$ if $\|x_n - x_{n-1}\| \leq t_n - t_{n-1}$, for all $n \in \mathbb{N}$.

Observe, from the last inequality, it follows the sequence $\{t_n\}$ is non-decreasing. The interest of the majorizing sequence is that the convergence of the sequence $\{x_n\}$ in the Banach space X is deduced from the convergence of the scalar sequence $\{t_n\}$, as we can see in the following result [12]:

Lemma 2. *Let $\{x_n\}$ be a sequence in a Banach space X and $\{t_n\}$ a majorizing sequence of $\{x_n\}$. Then, if $\{t_n\}$ converges to $t^* < \infty$, there exists $x^* \in X$ such that $x^* = \lim_n x_n$ and $\|x^* - x_n\| \leq t^* - t_n$, for $n = 0, 1, 2, \ldots$.*