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Tractability of approximation of ∞-variate functions with bounded mixed partial derivatives

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ABSTRACT

We study the tractability of ω-weighted L_s approximation for γ-weighted Banach spaces of ∞-variate functions with mixed partial derivatives of order r bounded in a ψ-weighted L_p norm. Functions from such spaces have a natural decomposition $f = \sum_{\mathfrak{u}} f_{\mathfrak{u}}$, where the summation is with respect to finite subsets $\mathfrak{u} \subset \mathbb{N}_+$ and each $f_{\mathfrak{u}}$ depends only on variables listed in $\mathfrak{u}$. We derive corresponding *multivariate decomposition methods* and show that they lead to polynomial tractability under suitable assumptions concerning γ weights as well as the probability density functions ω and ψ. For instance, suppose that the cost of evaluating functions with d variables is at most exponential in d and the weights γ decay to zero sufficiently quickly. Then the cost of approximating such functions with the error at most ε is proportional to $\varepsilon^{-1/(r+\min(1/s-1/p,0))}$ ignoring logarithmic terms. This is a nearly-optimal result, since (once again ignoring logarithmic terms) it equals the complexity of the same approximation problem in the univariate case.

1. Introduction

There are many practical problems dealing with ∞-variate functions. These involve stochastic differential equations, partial differential equations with random coefficients, and path integrals, see, e.g., [3,4,13,14,24] and papers cited therein.

This is why the study of complexity and tractability of such problems has become a popular field of research. Indeed, in addition to relatively early papers [11,21,29] on the complexity of such problems, the papers [1–6,9,12–14,17–20,24–27,31,32] have been written in the last three years.

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With the exception of [24–27,31,32] which study function approximation, the majority of the results are for approximating integrals. In the current paper, we focus on function approximation. As in the papers listed above, the space $\mathcal{F}$ is such that any $f \in \mathcal{F}$ has the unique representation

$$f(\boldsymbol{x}) = \sum_{\mathfrak{u}} f_{\mathfrak{u}}(\boldsymbol{x}),$$

where $\mathfrak{u}$ enumerate the finite subsets of $\mathbb{N}_+$ listing the so-called *active variables* upon which $f_{\mathfrak{u}}$ depends. More precisely, each $f_{\mathfrak{u}}$ belongs to a normed space $F_{\mathfrak{u}}$ that is a $|\mathfrak{u}|$-fold tensor product of a space F of univariate functions. Previous work on this topic assumed that F is a Hilbert space, with $F_{\mathfrak{u}}$ being Hilbert spaces obtained via the usual tensor product. In this paper, we let $F_{\mathfrak{u}}$ be Banach spaces of functions with mixed partial derivatives of order r bounded in a ψ-weighted L_p norm for some $p \in [1, \infty]$. The space $\mathcal{F}$ is endowed with the norm

$$\|f\|_{\mathcal{F}} = \left[\sum_{\mathfrak{u}} \left(\frac{\|f_{\mathfrak{u}}\|_{F_{\mathfrak{u}}}}{\gamma_{\mathfrak{u}}}\right)^q\right]^{1/q}$$

for $q \in [1, \infty]$ and a given family $\boldsymbol{\gamma} = \{\gamma_{\mathfrak{u}}\}_{\mathfrak{u}}$ of non-negative numbers, called *weights*. The goal is to approximate $f \in \mathcal{F}$ with the error measured in a ω-weighted $\mathcal{L}_s$-norm for some $s \in [1, \infty]$. Both ψ and ω are probability density functions.

The role of weights $\gamma_{\mathfrak{u}}$ has been explained in many papers; roughly speaking, they quantify the importance of interactions among variables listed in $\mathfrak{u}$. With the exception of [27], previous work has only considered the case $q = 2$, resulting in $\mathcal{F}$ also being a Hilbert space. In the current paper, we consider arbitrary q to study the tradeoff between the size of q and the rate of the decay of the weights $\boldsymbol{\gamma}$. In particular, we show that with $q = 1$, we have positive results even if $\gamma_{\mathfrak{u}}$ converges to zero very slowly. On the other hand, we need quickly-converging $\gamma_{\mathfrak{u}}$ for $q = \infty$. For more, see a short discussion at the end of the Introduction.

The role of ψ is to control the resulting space $\mathcal{F}$, since the faster the decay of ψ at $\pm\infty$ the larger the space $\mathcal{F}$. More precisely, the spaces $F_{\mathfrak{u}}$ are completions of $|\mathfrak{u}|$-times (algebraic) tensor products of the following space F of univariate functions. The domain of functions f from F is an arbitrary interval $D \subseteq \mathbb{R}$ and $f^{(r-1)}$ are absolutely (locally) continuous with bounded

$$\left[\int_D \left|f^{(r)}(x)\right|^p \psi(x)\,\mathrm{d}x\right]^{1/p} < \infty.$$

We prove the positive results in a constructive way by outlining the so-called *multivariate decomposition methods* (or *MDM* for short). These are modifications of methods introduced in [14], *changing dimension algorithms*. Roughly speaking, these methods identify a *set of important variable interactions* (or *SIVI* for short) $\mathfrak{U}(\varepsilon)$ with the following desirable properties:

(i) the cardinality of $\mathfrak{U}(\varepsilon)$ is polynomial in $1/\varepsilon$,
(ii) elements $f_{\mathfrak{u}}$ for $\mathfrak{u} \notin \mathfrak{U}(\varepsilon)$ can be neglected,
(iii) it is enough to approximate $f_{\mathfrak{u}}$ for $\mathfrak{u} \in \mathfrak{U}(\varepsilon)$,
(iv) the number $|\mathfrak{u}|$ of active variables is sub-logarithmic in $1/\varepsilon$ for all $\mathfrak{u} \in \mathfrak{U}(\varepsilon)$.

In other words, an MDM replaces one problem with ∞-many variables by a number of problems, each with at most $O(\ln(1/\varepsilon)/\ln(\ln(1/\varepsilon)))$ variables, which is very small.

In general, the terms $f_{\mathfrak{u}}$ in $f = \sum_{\mathfrak{u}} f_{\mathfrak{u}}$ are not available. Hence, to take advantage of (iii) and (iv), the space $\mathcal{F}$ has to be such that values of $f_{\mathfrak{u}}$ could be obtained by evaluating f at certain points. Moreover, we need efficient algorithms for multivariate problems with relatively small number $|\mathfrak{u}|$ of variables. When the $F_{\mathfrak{u}}$ are Hilbert spaces, Smolyak's construction provides such algorithms, see [23] for the idea and [28] for specific results. Such algorithms are often called *sparse grid algorithms*.

For Smolyak's construction to be applicable, one needs

$$\left\|\bigotimes_{j=1}^d \Delta_j\right\| \le \prod_{j=1}^d \|\Delta_j\|$$