Journal of Complexity 30 (2014) 347–354

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

An explicit construction of point sets with large minimum Dick weight

Kosuke Suzuki

Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo 153-8914, Japan

ARTICLE INFO

Article history:
Received 8 May 2013
Accepted 2 November 2013
Available online 11 December 2013

Keywords:
Quasi-Monte Carlo
Higher order digital net
Niederreiter–Xing sequence
WAFOM
Dick weight

ABSTRACT

The Dick weight is a generalization of the Hamming weight and the Niederreiter–Rosenbloom–Tsfasman (NRT) weight, defined on $\mathbb{F}_p^{s\times n}$. A point set with large value of minimum Dick weight gives a quadrature rule with small error in quasi-Monte Carlo integration. In this paper we explicitly construct point sets $P \subset \mathbb{F}_p^{s\times n}$ with large minimum Dick weight using Niederreiter–Xing sequences and Dick's interleaving construction. These point sets are also examples of low-WAFOM point sets.

1. Introduction

Let s, n be positive integers and p be a prime. Let $\mathbb{F}_2 = \{0, 1\}$ denote the two element field and $\mathbb{F}_p$ denote the p-element field. Let $V := \mathbb{F}_2^{s\times n}$ denote the set of $s \times n$ matrices with coefficients in $\mathbb{F}_2$, which is an sn-dimensional $\mathbb{F}_2$-vector space. For $A \in V$, we define the Dick weight as below.

Definition 1.1 ([4, *Definition 3.3*]). For $A = (a_{ij}) \in V$, we define a non-negative integer

$$\mu(A) := \sum_{\substack{1\le i\le s\\ 1\le j\le n}} j \times a_{i,j},$$

where each $a_{i,j} \in \{0, 1\}$ is considered as an integer, not as an element of $\mathbb{F}_2$. We call $\mu(A)$ the Dick weight (more precisely Dick ∞-weight) of A.

Remark 1.2. The original Dick weight is defined for $\mathbb{F}_2^{s\times \mathbb{N}}$: i.e. $n = \infty$ (see [1, Section 4.1] or [3, (15.2)]). This definition and Definition 3.2 are its truncated version up to n-digit precision.

E-mail address: ksuzuki@ms.u-tokyo.ac.jp.

0885-064X/$ – see front matter
http://dx.doi.org/10.1016/j.jco.2013.12.002

Let P be a nonzero linear subspace of V. We define the minimum Dick weight of $P \subset V$ by

$$\delta_P := \min_{A \in P \setminus \{\mathbf{0}\}} \mu(A).$$

Matsumoto and Yoshiki proved the existence of a subspace $P \subset V$ with $P^\perp$ having a large minimum Dick weight:

Proposition 1.3 ([5, Proposition 3]). *Let s, n be arbitrary positive integers. Let $\alpha := (\log 2)/4$ and assume $m \geq 4$. Then there exists a subspace $P \subset V$ of dimension m with $\delta_{P^\perp} \geq \alpha^2 m^2/s$.*

Here, we define $P^\perp := \{A \in \mathbb{F}_2^{s\times n} | A \cdot B = 0 \text{ for all } B \in P\}$, where the inner product on V is defined as usual: $A \cdot B = (a_{ij}) \cdot (b_{ij}) := \sum_{1\leq i\leq s, 1\leq j\leq n} a_{ij}b_{ij}$.

This shows that $\max\{\delta_{P^\perp} | P \subset \mathbb{F}_p^{s\times n}, \dim P = m\} \geq \alpha^2 m^2/s$. However, their proof of Proposition 1.3 uses a probabilistic argument and is therefore not constructive.

In this paper, using Niederreiter–Xing sequences and Dick's construction, we explicitly construct a linear subspace $P \subset V$ of dimension m which achieves $\delta_{P^\perp} \geq \lfloor m/11s \rfloor (m/2 + 8\sqrt{(s\lfloor m/11s \rfloor - 2)/3} + s/2 + 8) + 1$ when $s\lfloor m/11s \rfloor \geq 2$ for each m. This is the same order as m^2/s. This implies that we can explicitly construct point sets with low WAFOM.

Remark 1.4. Yoshiki [9, Lemma 1] proved that for any linear subspaces $P \subset V$ with $\dim P = m$, we have $\delta_{P^\perp} \leq (\frac{m}{2} + s)(\frac{m}{s} + 1)$. This inequality and the above results imply that $\max\{\delta_{P^\perp} | P \subset \mathbb{F}_2^{s\times n}, \dim P = m\}$ has the order m^2/s. He also proved that $\min\{\log(\mathrm{WAFOM}(P)) | P \subset \mathbb{F}_2^{s\times n}, \dim P = m\}$ has the order m^2/s.

The rest of this paper is organized as follows. In Section 2, we briefly provide some background on quasi Monte-Carlo and WAFOM. In Section 3.1, we recall the definition of higher order digital nets and Dick's construction. In Section 3.2, we recall results on Niederreiter–Xing sequences. In Section 4, we show our main results using Dick's construction and Niederreiter–Xing sequences.

2. Background

We explain on quasi-Monte Carlo (QMC) and WAFOM briefly (see [3] about QMC and [4] about WAFOM for details).

Let s, n be positive integers. Let $\mathcal{P} \subset [0, 1)^s$ be a point set in an s-dimensional unit cube with finite cardinality $|\mathcal{P}| = N$, and let $f : [0, 1)^s \to \mathbb{R}$ be an integrable function. Quasi-Monte Carlo integration by $\mathcal{P}$ is an approximation value

$$I_{\mathcal{P}}(f) := \frac{1}{N} \sum_{x \in \mathcal{P}} f(x)$$

of the actual integration

$$I(f) := \int_{[0,1)^s} f(x) dx.$$

The QMC integration error is $\mathrm{Err}(f; \mathcal{P}) := |I(f) - I_{\mathcal{P}}(f)|$.

We consider an n-digit discrete approximation. For a matrix $B := (b_{i,j}) \in V := \mathbb{F}_2^{s\times n}$ we associate an s-dimensional cube $\mathbf{I}_B := \prod_{i=1}^s I_{b_i} \subset [0, 1)^s$, where each edge $I_{b_i} := [\sum_{j=1}^n b_{i,j}2^{-j}, \sum_{j=1}^n b_{i,j}2^{-j} + 2^{-n})$ is a half-open interval with length 2^{-n}. We define an n-digit discrete approximation f_n of f as

$$f_n : \mathbb{F}_2^{s\times n} \to \mathbb{R}, \qquad B := (b_{i,j}) \mapsto \frac{1}{\mathrm{Vol}(\mathbf{I}_B)} \int_{\mathbf{I}_B} f(x) dx.$$

Let P be a subset of V. We define the n-th discretized QMC integration of f by P as

$$I_{P,n}(f) := \frac{1}{|P|} \sum_{B \in P} f_n(B),$$