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# Entropy numbers of convex hulls in Banach spaces and applications

Bernd Carl [a], Aicke Hinrichs [b,*], Philipp Rudolph [a]

[a] *Mathematisches Institut, Universität Jena, Ernst-Abbe-Platz 2, 07743 Jena, Germany*

[b] *Institut für Mathematik, Universität Rostock, Ulmenstraße 69, Haus 3, 18051 Rostock, Germany*

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ABSTRACT

In recent time much attention has been devoted to the study of entropy of convex hulls in Hilbert and Banach spaces and their applications in different branches of mathematics. In this paper we show how the rate of decay of the dyadic entropy numbers of a precompact set $A$ of a Banach space $X$ of type $p$, $1 < p \leq 2$, reflects the rate of decay of the dyadic entropy numbers of the absolutely convex hull $\mathrm{aco}(A)$ of $A$. Our paper is a continuation of the paper (Carl et al., 2013), where this problem has been studied in the Hilbert space case. We establish optimal estimates of the dyadic entropy numbers of $\mathrm{aco}(A)$ in the non-critical cases where the covering numbers $N(A, \varepsilon)$ of $A$ by $\varepsilon$-balls of $X$ satisfy the Lorentz condition

$$\int_0^\infty \left(\log_2(N(A,\varepsilon))\right)^{s/r} \mathrm{d}\varepsilon^s < \infty$$

for $0 < r < p'$, $0 < s < \infty$ or

$$\int_0^\infty \left(\log_2(2 + \log_2(N(A,\varepsilon)))\right)^{-\alpha s} \left(\log_2(N(A,\varepsilon))\right)^{s/r} \times \mathrm{d}\varepsilon^s < \infty$$

for $p' < r < \infty$, $0 < s \leq \infty$ and $\alpha \in \mathbb{R}$, with the usual modifications in the case $s = \infty$. The integral here is an improper Stieltjes integral and $p'$ is given by the Hölder condition $1/p + 1/p' = 1$. It turns out that, for fixed $s$, the entropy of the absolutely convex

* Corresponding author.
*E-mail addresses:* bernd.carl@uni-jena.de (B. Carl), aicke.hinrichs@uni-rostock.de (A. Hinrichs), philipp.rudolph@uni-jena.de (P. Rudolph).

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hull drastically changes if the parameter $r$ crosses the point $r = p'$. It is still an open problem what happens if $r = p'$ and $0 < s < \infty$. However, in the case $s = \infty$ we consider also the critical case $r = p'$ and, especially, the Hilbert space case $r = 2$.

We use the results for estimating entropy and Kolmogorov numbers of diverse operators acting from a Banach space whose dual space is of type $p$ or, especially, from a Hilbert space into a $C(M)$ space. In particular, we get entropy estimates of operators factoring through a diagonal operator and of abstract integral operators as well as of weakly singular convolution operators. Moreover, estimates of entropy and Kolmogorov numbers of the classical and generalized Riemann–Liouville operator are established, complementing and extending results in the literature.



## 1. Introduction

Let $(M, d)$ be a metric space and $B_M(x, \varepsilon) := \{y \in M \mid d(x, y) \leq \varepsilon\}$ the closed ball with center $x$ and radius $\varepsilon$. For a bounded subset $A$ of $M$ and a natural number $n \in \mathbb{N}$, the *nth entropy number of* $A$ is defined by

$$\varepsilon_n(A) := \inf\left\{\varepsilon \geq 0 \,\middle|\, \exists x_1, \ldots, x_n \in M : A \subset \bigcup_{i=1}^{n} B_M(x_i, \varepsilon)\right\}.$$

Moreover, the *nth dyadic entropy number* of $A$ is given by $e_n(A) := \varepsilon_{2^{n-1}}(A)$. The entropy numbers permit us to quantify precompactness, the rate of decay of $\varepsilon_n(A)$ can be interpreted as a degree of precompactness of the set $A$. In order to apply this idea to linear bounded operators $T \in \mathcal{L}(X, Y)$ between Banach spaces $X$ and $Y$ we define the *nth entropy number of* $T$ by

$$\varepsilon_n(T : X \to Y) := \varepsilon_n\big(T(B_X)\big),$$

where $B_X$ is the closed unit ball of $X$. The $n$th dyadic entropy number of $T$ is given by $e_n(T) := \varepsilon_{2^{n-1}}(T)$. According to the definition, the operator $T$ is compact if and only if $\varepsilon_n(T) \to 0$ for $n \to \infty$. The speed of decay of the entropy numbers of $T$ can be seen as a measure for the compactness of $T$. The concept of *covering numbers* is closely related to that of entropy numbers. For a bounded subset $A$ of a metric space $M$ and $\varepsilon > 0$ we define the $\varepsilon$-covering number $N(A, \varepsilon)$ by

$$N(A, \varepsilon) := \min\left\{n \in \mathbb{N} \,\middle|\, \exists x_1, \ldots, x_n \in M : A \subset \bigcup_{i=1}^{n} B_M(x_i, \varepsilon)\right\}.$$

In general, the absolutely convex hull $\operatorname{aco}(A)$ of a subset $A \subset X$ of a Banach space $X$ is much larger than $A$ itself. Nevertheless, if $A$ is precompact, then it is well-known that also $\operatorname{aco}(A)$ is precompact. Hence, it seems natural to ask how the rate of decay of the entropy numbers of $A$ reflects the rate of decay of the entropy numbers of $\operatorname{aco}(A)$. In recent years this problem has been intensively studied in different settings (cf. e.g. [2,9–16,18,21,27–29,33–36,39,41,43,46,59,64–66]). In our setting, we consider a precompact subset $A$ of a Banach space of type $p$, $1 < p \leq 2$. We assume that the dyadic entropy numbers of $A$ belong to a generalized Lorentz sequence space, which implies certain decay or summability properties. These spaces will play a key role in our further considerations, because they allow a unified formulation of summability properties and decay rates. In the following, let $x = (\xi_n)_{n=1}^{\infty}$ be a non-increasing bounded sequence of non-negative real numbers. For given $0 < r < \infty$, $0 < s \leq \infty$ and $\alpha \in \mathbb{R}$ we say that $x$ belongs to the generalized Lorentz sequence space $l_{r,s,\alpha}$ if

$$\left((\log(n+1))^{-\alpha}\, n^{1/r-1/s}\, \xi_n\right)_{n=1}^{\infty} \in l_s,$$