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Tractability results for the weighted star-discrepancy

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ABSTRACT

The weighted star-discrepancy has been introduced by Sloan and Woźniakowski to reflect the fact that in multidimensional integration problems some coordinates of a function may be more important than others. It provides upper bounds for the error of multidimensional numerical integration algorithms for functions belonging to weighted function spaces of Sobolev type. In the present paper, we prove several tractability results for the weighted star-discrepancy. In particular, we obtain rather sharp sufficient conditions under which the weighted star-discrepancy is strongly tractable. The proofs are probabilistic, and use empirical process theory.

1. Introduction

For a set of points $x_1, \ldots, x_N$ from the d-dimensional unit cube $[0, 1]^d$, for any $z \in [0, 1]^d$ the *discrepancy function* $\Delta(z)$ is defined as

$$\Delta(z) = \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[0,z)}(x_n) - \lambda([0, z)),$$

and the star-discrepancy $D_N^*(x_1, \ldots, x_N)$ is defined as

$$D_N^*(x_1, \ldots, x_N) = \sup_{z \in [0,1]^d} |\Delta(z)| .$$

Here $[0, z)$ is an axis-parallel box that stretches from the origin to z, and λ denotes the (d-dimensional) Lebesgue measure. The Koksma–Hlawka inequality states that for a function f on $[0, 1]^d$ the difference

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between the arithmetic mean of the function values $f(x_1), \ldots, f(x_N)$ and the integral of f over $[0, 1]^d$ is bounded by the star-discrepancy of $x_1, \ldots, x_N$, multiplied with the (Hardy–Krause) variation of f over $[0, 1]^d$. Consequently, point sets having small star-discrepancy can be used to approximate a multidimensional integral. This method for numerical integration is an example of the so-called *quasi-Monte Carlo (QMC) method*, which uses cleverly constructed deterministic point sets as sampling points (as opposed to the Monte Carlo method, where randomly sampled points are used).

There exist several constructions of point sets achieving a discrepancy of order $\ll (\log N)^{d-1} N^{-1}$, for fixed d and for $N \to \infty$. However, these bounds are only useful if the number of points N is very large (i.e., at least exponential) in comparison with d, which means that QMC integration using such points is not feasible on a computer if d is large. To describe the problem concerning the existence of low-discrepancy point sets of moderate cardinality for large values of d, the notion of the *inverse of the star-discrepancy* can be used. Let $n^*(d, \varepsilon)$ denote the smallest possible cardinality of a point set in $[0, 1]^d$ having discrepancy at most ε. By a result of Heinrich et al. [11] for any d and N there exist points $x_1, \ldots, x_N \in [0, 1]^d$ such that

$$D_N^*(x_1, \ldots, x_N) \le c_{\text{abs}} \frac{\sqrt{d}}{\sqrt{N}} \tag{1}$$

(where we can choose $c_{\text{abs}} = 10$, see [1]), which implies that

$$n^*(d, \varepsilon) \le c_{\text{abs}} d \varepsilon^{-2}$$

(c_{abs} denotes positive absolute constants, not always the same). On the other hand, Hinrichs [14] proved the lower bound

$$n^*(d, \varepsilon) \ge c_{\text{abs}} d \varepsilon^{-1}.$$

Thus there exist high-dimensional low-discrepancy point sets which have moderate cardinality in comparison with the dimension d. Note, however, that constructing such point sets is a largely unsolved problem (cf. [7,8]), and that calculating (or estimating) the discrepancy of a given high-dimensional point set is generally a very difficult problem (see [9]).

A series of numerical investigations of Paskov and Traub in the mid-1990s showed that in practice QMC integration can still be successfully applied to high-dimensional problems, and often perform significantly better than what could be expected from theoretical upper bounds (see [22]). One possible explanation is that often for a formally high-dimensional problem only a small number of coordinates is really important, while other (or most) coordinates are much less important. This idea led to the introduction of weighted function spaces and weighted discrepancies by Sloan and Woźniakowski [23]. These concepts are closely connected with the theory of (weighted) reproducing kernel Hilbert spaces of Sobolev type; in particular, the error of a QMC integration scheme for a function f from such a weighted space can be estimated in terms of the norm of f in this space and the corresponding weighted discrepancy of the set of sampling points, by means of a weighted Koksma–Hlawka inequality. For details, see [23] as well as [3,6].

By the expression *weights* we mean a set γ of non-negative real numbers $\gamma_{\mathfrak{u}}$, indexed by the class of all non-empty subsets $\mathfrak{u}$ of the set of coordinates $\{1, \ldots, d\}$ (or indexed by the class of all non-empty subsets of $\mathbb{N}$). An important special case is *product weights*, which satisfy

$$\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j,$$

where γ_j is the weight of $\{j\}$, that is, the weight associated with the jth coordinate.

Let $|\mathfrak{u}|$ denote the cardinality of $\mathfrak{u}$. For a point $x \in [0, 1]^d$ and a non-empty subset $\mathfrak{u}$ of $\{1, \ldots, d\}$, we write $x(\mathfrak{u})$ for the $|\mathfrak{u}|$-dimensional point which consists only of those coordinates of x whose index belongs to $\mathfrak{u}$. Furthermore, we write $(x(\mathfrak{u}); 1)$ for the d-dimensional vector which has the same coordinates as x, except that coordinates whose index is not in $\mathfrak{u}$ are replaced by 1. Then the *weighted star-discrepancy* of the points $x_1, \ldots, x_N \in [0, 1]^d$ for weights $\gamma = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \{1, \ldots, d\}}$ is defined as

$$D_{N,\gamma}^*(x_1, \ldots, x_N) = \sup_{z \in [0,1]^d} \max_{\mathfrak{u} \subset \{1, \ldots, d\}} \gamma_{\mathfrak{u}} |\Delta(z(\mathfrak{u}); 1)|.$$