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Two-step Newton methods

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ABSTRACT

We present sufficient convergence conditions for two-step Newton methods in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. The advantages of our approach over other studies such as Argyros et al. (2010) [5], Chen et al. (2010) [11], Ezquerro et al. (2000) [16], Ezquerro et al. (2009) [15], Hernández and Romero (2005) [18], Kantorovich and Akilov (1982) [19], Parida and Gupta (2007) [21], Potra (1982) [23], Proinov (2010) [25], Traub (1964) [26] for the semilocal convergence case are: weaker sufficient convergence conditions, more precise error bounds on the distances involved and at least as precise information on the location of the solution. In the local convergence case more precise error estimates are presented. These advantages are obtained under the same computational cost as in the earlier stated studies. Numerical examples involving Hammerstein nonlinear integral equations where the older convergence conditions are not satisfied but the new conditions are satisfied are also presented in this study for the semilocal convergence case. In the local case, numerical examples and a larger convergence ball are obtained.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

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where, F is a Fréchet-differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$. Many problems in Applied Sciences reduce to solving an equation in the form (1.1). These solutions can be rarely found in closed form. That is why the most solution methods for these equations are iterative. The convergence analysis of iterative methods is usually divided into two categories: semilocal and local convergence analysis. In the semilocal convergence analysis one derives convergence criteria from the information around an initial point whereas in the local analysis one finds estimates of the radii of convergence balls from the information around a solution.

The Newton method defined by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad \text{for each } n = 0, 1, 2, \dots, \tag{1.2}$$

where x_0 is an initial point, is undoubtedly the most popular iterative method for generating a sequence approximating $x^\star$. The Newton method is quadratically convergent if x_0 is chosen sufficiently close to the solution $x^\star$. There is a plethora of local as well as semilocal convergence results for the Newton method. We refer the reader to [1–26] (and the references there in) for the history and recent results on the Newton method. In order to increase the convergence order higher convergence order iterative methods have also been used [1,3,5–7,9,11,14–18,21,22,26,27]. The convergence domain usually gets smaller as the order of convergence of the method increases. That is why it is important to enlarge the convergence domain as much as possible using the same conditions and constants as before. This is our main motivation for this paper. In particular, we revisit the two-step Newton methods defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - F'(y_n)^{-1}F(y_n) \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - F'(x_n)^{-1}F(y_n). \end{aligned} \tag{1.4}$$

Two-step Newton methods (1.3) and (1.4) are of convergence order four and three, respectively [1,3,6,7,15,18]. It is well known that if the Lipschitz condition

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \le L\|x - y\| \quad \text{for each } x \text{ and } y \in \mathcal{D} \tag{1.5}$$

as well as

$$\|F'(x_0)^{-1}F(x_0)\| \le \nu \tag{1.6}$$

holds for some $L > 0$ and $\nu > 0$, then the sufficient semilocal convergence condition for both the Newton method (1.2) and the two-step Newton method (1.3) is given by the famous, for its simplicity and clarity, Newton–Kantorovich hypothesis [19]:

$$h = L\nu \le \frac{1}{2}. \tag{1.7}$$

Hypothesis (1.7) is only sufficient for the convergence of the Newton method. That is why we challenged it in a series of papers [1–8] by introducing the center-Lipschitz condition

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le L_0\|x - x_0\| \quad \text{for each } x \in \mathcal{D}. \tag{1.8}$$

Notice that

$$L_0 \le L \tag{1.9}$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large [2,3,6,8]. Our sufficient convergence conditions are given by

$$h_1 = L_1\nu \le \frac{1}{2}, \tag{1.10}$$

$$h_2 = L_2\nu \le \frac{1}{2}, \tag{1.11}$$