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Approximation of analytic functions in Korobov spaces

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ABSTRACT

We study multivariate L_2-approximation for a weighted Korobov space of analytic periodic functions for which the Fourier coefficients decay exponentially fast. The weights are defined, in particular, in terms of two sequences $\boldsymbol{a} = \{a_j\}$ and $\boldsymbol{b} = \{b_j\}$ of positive real numbers bounded away from zero. We study the minimal worst-case error $e^{L_2-\mathrm{app},\Lambda}(n, s)$ of all algorithms that use n information evaluations from the class Λ in the s-variate case. We consider two classes Λ in this paper: the class Λ^{all} of all linear functionals and the class Λ^{std} of only function evaluations.

We study exponential convergence of the minimal worst-case error, which means that $e^{L_2-\mathrm{app},\Lambda}(n, s)$ converges to zero exponentially fast with increasing n. Furthermore, we consider how the error depends on the dimension s. To this end, we define the notions of weak, polynomial and strong polynomial tractability. In particular, polynomial tractability means that we need a polynomial number of information evaluations in s and $1 + \log \varepsilon^{-1}$ to compute an ε-approximation. We derive necessary and sufficient conditions on the sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ for obtaining exponential error convergence, and also for obtaining the various notions of tractability. The results are the same for both classes Λ. They are also constructive with the exception of one particular subcase for which we provide a semi-constructive algorithm.

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1. Introduction

We study approximation of s-variate functions defined on the unit cube $[0, 1]^s$ with the worst-case error measured in the L_2 norm. Multivariate approximation is a problem that has been studied in a vast number of papers from many different perspectives. We consider analytic periodic functions belonging to a weighted Korobov space. We present necessary and sufficient conditions on the decay of the Fourier coefficients under which we can achieve exponential and uniform exponential convergence with various notions of tractability.

We approximate functions by algorithms that use n information evaluations. We either allow information evaluations from the class Λ^{all} of all continuous linear functionals or from the class Λ^{std} of standard information which consists of only function evaluations.

For large s, it is important to study how the errors of algorithms depend not only on n but also on s. The information complexity $n^{L_2\text{-app},\Lambda}(\varepsilon, s)$ is the minimal number n for which there exists an algorithm using n information evaluations from the class $\Lambda \in \{\Lambda^{\mathrm{all}}, \Lambda^{\mathrm{std}}\}$ with an error at most ε in the s-variate case. The information complexity is proportional to the minimal cost of computing an ε-approximation since linear algorithms are optimal and their cost is proportional to $n^{L_2\text{-app},\Lambda}(\varepsilon, s)$.

We would like to control how $n^{L_2\text{-app},\Lambda}(\varepsilon, s)$ depends on ε^{-1} and s. In the standard study of tractability, see [6–8], *weak tractability* means that $n^{L_2\text{-app},\Lambda}(\varepsilon, s)$ is *not* exponentially dependent on ε^{-1} and s. Furthermore, *polynomial tractability* means that $n^{L_2\text{-app},\Lambda}(\varepsilon, s)$ is polynomially bounded by $C\, s^q\, \varepsilon^{-p}$ for some C, q and p independent of $\varepsilon \in (0, 1)$ and $s \in \mathbb{N}$. If $q = 0$ then we have *strong polynomial tractability*.

Typically, $n^{L_2\text{-app},\Lambda}(\varepsilon, s)$ is polynomially dependent on ε^{-1} and s for weighted classes of smooth functions. The notion of weighted function classes means that the successive variables and groups of variables are moderated by certain weights. For sufficiently fast decaying weights, the information complexity depends at most polynomially on s, and we obtain polynomial tractability, or even strong polynomial tractability.

These notions of tractability are suitable for problems for which smoothness of functions is finite. This means that functions are differentiable only finitely many times. Then the minimal errors of algorithms enjoy polynomial convergence and are bounded by $C(s)\, n^{-\tau}$, for some positive $C(s)$ which depends only on s and some positive τ which depends on the smoothness of functions and may also depend on s. For many classes of such functions we know the largest τ which grows with increasing smoothness and decreasing weights. Furthermore, if τ is independent of s, weak tractability holds if $\log C(s) = o(s)$, whereas polynomial tractability holds if $C(s)$ is polynomially dependent on s, and strong polynomial tractability holds if $C(s)$ is uniformly bounded in s.

It seems to us that the case of analytic or infinitely many times differentiable functions is also of interest. For such classes of functions we would like to replace polynomial convergence by exponential convergence, and study the same notions of tractability in terms of $(1 + \log\, \varepsilon^{-1}, s)$ instead of (ε^{-1}, s). More precisely, let $e^{L_2\text{-app},\Lambda}(n, s)$ be the minimal worst-case error among all algorithms that use n information evaluations from a permissible class Λ in the s-variate case. By exponential convergence of the nth minimal approximation error we mean that

$$e^{L_2\text{-app},\Lambda}(n, s) \le C(s)\, q^{\,(n/C_1(s))^{p(s)}} \quad \text{for all } n, s \in \mathbb{N}.$$

Here, $q \in (0, 1)$ is independent of s, whereas C, C_1, and p are allowed to be dependent on s. We speak of uniform exponential convergence if p can be replaced by a positive number independent of s. A priori it is not obvious what we should require about $C(s)$, $C_1(s)$ and $p(s)$ although, clearly, the smaller $C(s)$ and $C_1(s)$ the better, and we would like to have $p(s)$ as large as possible. Obviously, if we do not care about the dependence on s then the mere existence of $C(s)$, $C_1(s)$ and $p(s)$ is enough.

The last bound on $e^{L_2\text{-app},\Lambda}(n, s)$ yields

$$n^{L_2\text{-app},\Lambda}(\varepsilon, s) \le \left\lceil C_1(s) \left(\frac{\log C(s) + \log \varepsilon^{-1}}{\log q^{-1}} \right)^{1/p(s)} \right\rceil \quad \text{for all } s \in \mathbb{N} \text{ and } \varepsilon \in (0, 1).$$

Exponential convergence implies that asymptotically with respect to ε tending to zero, we need $\mathcal{O}(\log^{1/p(s)} \varepsilon^{-1})$ information evaluations to compute an ε-approximation to functions from the Korobov space. (Throughout the paper log means the natural logarithm and $\log^r x$ means $[\log x]^r$.)