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On weighted Hilbert spaces and integration of functions of infinitely many variables

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ABSTRACT

We study aspects of the analytic foundations of integration and closely related problems for functions of infinitely many variables $x_1, x_2, \ldots \in D$. The setting is based on a reproducing kernel k for functions on D, a family of non-negative weights γ_u, where u varies over all finite subsets of $\mathbb{N}$, and a probability measure ρ on D. We consider the weighted superposition $K = \sum_u \gamma_u k_u$ of finite tensor products k_u of k. Under mild assumptions we show that K is a reproducing kernel on a properly chosen domain in the sequence space $D^{\mathbb{N}}$, and that the reproducing kernel Hilbert space $H(K)$ is the orthogonal sum of the spaces $H(\gamma_u k_u)$. Integration on $H(K)$ can be defined in two ways, via a canonical representer or with respect to the product measure $\rho^{\mathbb{N}}$ on $D^{\mathbb{N}}$. We relate both approaches and provide sufficient conditions for the two approaches to coincide.

Dedicated to J.F. Traub and G.W. Wasilkowski on the occasion of their 80th and 60th birthdays

1. Introduction

For functions of infinitely many variables $x_1, x_2, \ldots \in D$ with D denoting a non-empty set, the study of quadrature problems and their complexity was initiated in [10], and it has intensively been studied recently, see [2,6,8,9,15–17,20] and the preprints [3,4,7]. In the same setting function approximation

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is studied in [23,25,26], linear tensor product problems are studied in [24], and a non-linear problem associated with elliptic PDEs with random coefficients is studied in [13,14]. See [22] for a survey.

The present paper is devoted to some aspects of the analytic foundations of computational problems of this kind.

Let us outline the setting in the references mentioned above together with a discussion of our results. At first we consider the underlying function spaces, and then we turn to the integration functional, which is to be approximated.

The construction of spaces of functions with an infinite number of variables is based on a reproducing kernel k for functions of a single variable $x \in D$ and on a family of weights $\gamma_u \geq 0$, which indicate the importance of the group $(x_j)_{j\in u}$ of variables for finite sets $u \subseteq \mathbb{N}$. Formally, this leads to

$$K = \sum_u \gamma_u k_u,$$

where u varies over all finite subsets of $\mathbb{N}$ and where k_u is the $|u|$-fold tensor product of k such that the functions in the associated reproducing kernel Hilbert space $H(k_u)$ only depend on $(x_j)_{j\in u}$.

In this paper we study a domain $\mathfrak{X} \subseteq D^{\mathbb{N}}$ such that K is actually a reproducing kernel on $\mathfrak{X} \times \mathfrak{X}$ and that the spaces $H(\gamma_u k_u)$ form an orthogonal decomposition of $H(K)$ under mild assumptions. The latter fact has been used in [9,10], e.g., without providing a rigorous proof. Moreover, we show that the space $H(K)$ is isometrically isomorphic in a natural way to the quasi-reproducing kernel Hilbert space introduced and studied in [15,20,25,26] for integration and function approximation.

Two different ways are used to define the integration functional for $f \in H(K)$. Both of these constructions are based on a probability measure ρ on D such that $H(k) \subseteq L_1(\rho)$, which implies

$$\int_D g \, d\rho = \langle g, h \rangle_k$$

for every $g \in H(k)$ with a representer $h \in H(k)$.

Either, one studies the Lebesgue integral with respect to the product measure $\mu = \rho^{\mathbb{N}}$ on $D^{\mathbb{N}}$. Taking into account that $\mu(\mathfrak{X}) \in \{0, 1\}$, the functions $f \in H(K)$ have to be properly extended from $\mathfrak{X}$ to $D^{\mathbb{N}}$, in particular if $\mu(\mathfrak{X}) = 0$. At this point we are free to think of the kernels k_u as being defined on $\mathfrak{X} \times \mathfrak{X}$ or $D^u \times D^u$. This distinction is indeed only of a technical nature, which will become clear when we introduce the kernels rigorously. The extension Tf is given as the L_1-limit of the orthogonal projections of f onto the spaces $H(\sum_{u\subseteq\{1,\ldots,s\}} \gamma_u k_u)$, and it leads to the integral

$$I_1(f) = \int_{D^{\mathbb{N}}} Tf \, d\mu.$$

Clearly

$$I_1(f) = \int_{\mathfrak{X}} f \, d\mu$$

if $\mu(\mathfrak{X}) = 1$. Cf. [2–4,6,8,7,9,10,17].

Alternatively, one studies the bounded linear functional

$$I_2(f) = \left\langle f, \sum_u \gamma_u h_u \right\rangle_K$$

on $H(K)$, where

$$h_u(\mathbf{x}) = \prod_{j\in u} h(x_j), \quad \mathbf{x} \in \mathfrak{X}.$$

Here, as previously, u varies over all finite subsets of $\mathbb{N}$. We are free to think of h_u as being defined on $\mathfrak{X}$ or D^u, so that this function is the representer of integration with respect to the product measure ρ^u on D^u. Cf. [15,20].