Journal of Complexity 30 (2014) 48–55

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

Weak and quasi-polynomial tractability of approximation of infinitely differentiable functions

Jan Vybíral

Technical University Berlin, Department of Mathematics, Secretary Office MA 4-1, Street of 17. June 136, 10623 Berlin, Germany

ARTICLE INFO

Article history:
Available online 12 April 2013

Dedicated to J.F. Traub and G.W. Wasilkowski on the occasion of their 80th and 60th birthdays

Keywords:
Weak tractability
Uniform approximation
Infinitely differentiable functions
Curse of dimensionality

ABSTRACT

We comment on recent results in the field of information based complexity, which state (in a number of different settings), that the approximation of infinitely differentiable functions is intractable and suffers from the curse of dimensionality. We show that renorming the space of infinitely differentiable functions in a suitable way allows weakly tractable uniform approximation by using only function values. Moreover, the approximating algorithm is based on a simple application of Taylor's expansion about the center of the unit cube. We discuss also the approximation on the Euclidean ball and the approximation in the L_1-norm.

1. Introduction

We consider different classes F_d of infinitely-differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ and discuss algorithms using only function values of f in order to approximate f uniformly, or in the L_1-norm. We are especially interested in the case of large $d \gg 1$.

In the classical setting of approximation theory, the dimension of the Euclidean space d is fixed. Furthermore, the decay of the minimal error $e(n)$ of approximation of smooth functions in a Lebesgue space norm is very well studied for both algorithms using n arbitrary linear functionals and for algorithms using only n function evaluations. We refer to [5–7,12,14,16,17] and references therein. The decay is usually polynomial, speeds up with increasing smoothness and slows down with increasing dimension. Furthermore, this terminology typically hides the dependence of the constants on the dimension d, which might be even exponential. This motivates the question of what happens if both the dimension d and the smoothness parameter s tend to infinity.

E-mail address: vybiral@math.tu-berlin.de.

0885-064X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jco.2013.04.003

If $n(\varepsilon, d)$ denotes the minimal number of function values needed to approximate all functions from F_d up to the error $\varepsilon > 0$, we say that the problem suffers from the *curse of dimensionality*, if $n(\varepsilon, d)$ grows exponentially in d. This means, that there are positive numbers c, ε_0 and γ, such that

$$n(\varepsilon, d) \geq c(1+\gamma)^d \quad \text{for all } 0 < \varepsilon \leq \varepsilon_0 \text{ and infinitely many } d \in \mathbb{N}.$$

On the other hand, we say that the problem is *weakly tractable* if

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, d)}{\varepsilon^{-1}+d} = 0.$$

Furthermore, the problem is *quasi-polynomially tractable* in the sense of [1], if there exist two constants $C, t > 0$, such that

$$n(\varepsilon, d) \leq C \exp\{t(1+\ln(1/\varepsilon))(1+\ln(d))\} \tag{1}$$

for all $0 < \varepsilon < 1$ and all $d \in \mathbb{N}$. For the sake of completeness, we add that a problem is *polynomially tractable* if there exist non-negative numbers C, p and q such that

$$n(\varepsilon, d) \leq C\varepsilon^{-p} d^q \quad \text{for all } 0 < \varepsilon < 1 \text{ and } d \in \mathbb{N}.$$

If $q = 0$ above then a problem is *strongly polynomially tractable*. We refer to the monographs [15,8,11] for a detailed discussion of these and other kinds of *(in)tractability* and the closely related field of *information based complexity*.

The L_∞-approximation of infinitely differentiable functions was studied in [4], where the authors showed that the problem is not strongly polynomially tractable. It was also discussed in [8], cf. Open Problem 2 therein. An essential breakthrough was achieved in [9] (which in turn is based on [10] and answers an open problem posed there), where uniform approximation of the functions from the class

$$\mathbb{F}_d = \left\{ f : [0,1]^d \to \mathbb{R} : \sup_{\alpha\in\mathbb{N}_0^d} \|D^\alpha f\|_\infty \leq 1 \right\}$$

was shown to satisfy $n(\varepsilon, d) \geq 2^{\lfloor d/2 \rfloor}$ for all $0 < \varepsilon < 1$ and all $d \in \mathbb{N}$ and this result is also true if arbitrary linear functionals are allowed as the information map about f. Hence, the problem is intractable and suffers from the curse of dimensionality. In the context of weighted spaces of infinitely differentiable functions, the problem was also discussed in [18].

Multivariate integration of infinitely differentiable functions from the class $\mathbb{F}_d$ was conjectured not to be polynomially tractable in [20] and was shown not to be strongly polynomially tractable in [19]. Furthermore, it is known (cf. [13,2]) that multivariate integration of functions from

$$C_d^k = \left\{ f : [0,1]^d \to \mathbb{R} : \sup_{\alpha:|\alpha|\leq k} \|D^\alpha f\|_\infty \leq 1 \right\}$$

suffers from the curse of dimensionality for all $k \in \mathbb{N}$. Although multivariate integration of infinitely differentiable functions is also discussed in [2,3], it seems to be still an open problem if the curse of dimensionality holds also for multivariate integration and the class $\mathbb{F}_d$.

The main result of this paper is the following.

Theorem 1. (i) *Uniform approximation on the cube* $[-1/2, 1/2]^d$ *of functions from the class*

$$F_d^1 = \left\{ f \in C^\infty([-1/2,1/2]^d) : \sup_{k\in\mathbb{N}_0} \sum_{|\beta|=k} \frac{\|D^\beta f\|_\infty}{\beta!} \leq 1 \right\} \tag{2}$$

is quasi-polynomially tractable.

(ii) *Uniform approximation on the balls* $B(0, r_d) = \{x \in \mathbb{R}^d : \|x\|_2 \leq r_d\}$, *where* r_d *are chosen in such a way, that the volume of* $B(0, r_d)$ *is equal to one, and the functions are from the class*

$$F_d^2 = \left\{ f \in C^\infty(B(0, r_d)) : \sup_{k\in\mathbb{N}_0} \|\partial_\nu^k f\|_\infty \leq 1 \right\} \tag{3}$$

is weakly tractable. Here, $(\partial_\nu^k f)(x)$ *denotes the k-th derivative of f at* $x \neq 0$ *in the "normal" direction* $x/\|x\|_2$.