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Approximation numbers of Sobolev embeddings—Sharp constants and tractability

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ABSTRACT

We investigate optimal linear approximations (approximation numbers) in the context of periodic Sobolev spaces $H^s(\mathbb{T}^d)$ of fractional smoothness $s > 0$ for various equivalent norms including the classical one. The error is always measured in $L_2(\mathbb{T}^d)$. Particular emphasis is given to the dependence of all constants on the dimension d. We capture the exact decay rate in n and the exact decay order of the constants with respect to d, which is in fact polynomial. As a consequence we observe that none of our considered approximation problems suffers from the curse of dimensionality. Surprisingly, the square integrability of all weak derivatives up to order three (classical Sobolev norm) guarantees weak tractability of the associated multivariate approximation problem.

1. Introduction

In the present paper, we investigate the asymptotic behavior of the approximation numbers of the embeddings

$$I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d), \quad s > 0, d \in \mathbb{N}, \tag{1.1}$$

where $H^s(\mathbb{T}^d)$ is the periodic Sobolev space of fractional smoothness $s > 0$ on the d-torus. The approximation numbers of a bounded linear operator $T : X \to Y$ between two Banach spaces are

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defined as

$$a_n(T : X \to Y) := \inf_{\operatorname{rank} A < n} \sup_{\|x|X\| \le 1} \| Tx - Ax|Y\| = \inf_{\operatorname{rank} A < n} \|T - A : X \to Y\|, \quad n \in \mathbb{N}. \tag{1.2}$$

They describe the best approximation of T by finite rank operators. If X and Y are Hilbert spaces and T is compact, then $a_n(T)$ is the nth singular number of T.

The first result on the approximation of Sobolev embeddings is due to Kolmogorov [5]. He showed already in 1936 that in the univariate (homogeneous) case with integer smoothness $m \in \mathbb{N}$ the approximation numbers $a_n(I_d : \dot{H}^m(\mathbb{T}) \to L_2(\mathbb{T}))$ decay exactly like n^{-m}. Here we are interested in the multivariate (inhomogeneous) situation, where d is large, and investigate the approximation numbers $a_n(I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ for arbitrary smoothness parameters $s > 0$.

In fact, there is an increasing interest in the approximation of multivariate functions since many problems from, e.g., finance or quantum chemistry, are modeled in associated function spaces on high-dimensional domains. So far, many authors have contributed to the subject, see for instance the monographs by Temlyakov [16] and Tikhomirov [17] and the references therein. In [16, Chapter 2, Theorems 4.1, 4.2] the following two-sided estimate can be found

$$c_s(d)\, n^{-s/d} \le a_n(I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le C_s(d)\, n^{-s/d}, \quad n \in \mathbb{N},$$

where the constants $c_s(d)$ and $C_s(d)$, only depending on d and s, were not explicitly determined. Our main focus is to clarify, for arbitrary but fixed $s > 0$, the dependence of these constants on d. Surprisingly, it turns out that the optimal constants decay polynomially in d, i.e.,

$$c_s(d) \sim C_s(d) \sim d^{-\alpha} \tag{1.3}$$

for some $\alpha > 0$ which depends on the chosen norm in $H^s(\mathbb{T}^d)$ and the value of the smoothness parameter $s > 0$. We give exact values of α in at least two important situations.

As a consequence of these precise estimates for the approximation numbers we obtain weak tractability results for the approximation problem of the Sobolev embeddings (1.1). Basically, we consider three different (but of course) equivalent norms on $H^s(\mathbb{T}^d)$, see (2.6)–(2.8) below. The first two norms are the most common natural norms obtained by taking distributional derivatives in the case s being an integer. It turns out that all the associated approximation problems do not suffer from the curse of dimensionality. In fact, we even obtain weak tractability in some of the important cases, i.e., if the smoothness s is larger than one or two, respectively, depending on the used norm. This is a quite surprising fact when taking the famous negative result into account that the approximation of infinitely differentiable functions is intractable [10]. See Remark 5.8 below for a more detailed comparison. In the case of Sobolev smoothness and L_2-approximation it seems that already less smoothness restrictions guarantee weak tractability in the worst case setting. Furthermore, our results illustrate that the notion of tractability is sensitive with respect to the choice of the equivalent norms.

The paper is organized as follows. In Section 2 we recall the definition of periodic Sobolev spaces $H^s(\mathbb{T}^d)$ and discuss various equivalent norms. In addition, we will recall some facts on Hilbert spaces, diagonal operators and associated approximation numbers. Section 3 is devoted to provide some useful combinatorial identities and related inequalities. Section 4 is the heart of this paper. Here we prove estimates of the approximation numbers as indicated above. In the final Section 5 we apply the obtained results to establish results on weak tractability.

Notation. As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. With $\mathbb{T}$ we denote the torus represented by the interval $[0, 2\pi]$. For a real number a we put $a_+ := \max\{a, 0\}$. The symbol d is always reserved for the dimension in $\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{N}^d$, and $\mathbb{T}^d$. For $0 < p \le \infty$ and $x \in \mathbb{R}^d$ we denote $|x|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$ with the usual modification in the case $p = \infty$. If X and Y are two Banach spaces, the norm of an element x in X will be denoted by $\|x|X\|$ and the norm of an operator $A : X \to Y$ is denoted by $\|A : X \to Y\|$. The symbol $X \hookrightarrow Y$ indicates that the embedding operator is continuous.