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The curse of dimensionality for numerical integration of smooth functions II

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ABSTRACT

We prove the curse of dimensionality in the worst case setting for numerical integration for a number of classes of smooth d-variate functions. Roughly speaking, we consider different bounds for the directional or partial derivatives of $f \in C^k(D_d)$ and ask whether the curse of dimensionality holds for the respective classes of functions. We always assume that $D_d \subset \mathbb{R}^d$ has volume one and we often assume additionally that D_d is either convex or that its radius is proportional to $\sqrt{d}$. In particular, D_d can be the unit cube. We consider various values of k including the case $k = \infty$ which corresponds to infinitely differentiable functions. We obtain necessary and sufficient conditions, and in some cases a full characterization for the curse of dimensionality. For infinitely differentiable functions we prove the curse if the bounds on the successive derivatives are appropriately large. The proof technique is based on a volume estimate of a neighborhood of the convex hull of n points which decays exponentially fast in d. For $k = \infty$, we also study conditions for quasi-polynomial, weak and uniform weak tractability. In particular, weak tractability holds if all directional derivatives are bounded by one. It is still an open problem if weak tractability holds if all partial derivatives are bounded by one.

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1. Introduction

We study the problem of numerical integration, i.e., of approximating the integral

$$S_d(f) = \int_{D_d} f(x)\,\mathrm{d}x \tag{1}$$

over an open subset $D_d \subset \mathbb{R}^d$ of Lebesgue measure $\lambda_d(D_d) = 1$ for integrable functions $f\colon D_d \to \mathbb{R}$. In particular, we consider the case of *smooth* integrands. The main interest is on the behavior of the minimal number of function values that are needed in the worst case setting to achieve an error at most $\varepsilon > 0$, while the dimension d tends to infinity. Note that classical examples of domains D_d are the unit cube $[0, 1]^d$ and the normalized Euclidean ball (with volume 1), which are closed. However, we work with their interiors for definiteness of certain derivatives. Obviously, this does not change the integration problem.

We always consider sets D_d for which $\lambda_d(D_d) = 1$. This assumption guarantees that the integration problem is properly normalized and suffices to establish the curse of dimensionality for a number of classes considered in this paper. To obtain necessary and sufficient conditions on the curse, we need further assumptions on D_d. Typically we assume that D_d is the unit cube or that D_d is convex or that D_d satisfies property (P) which roughly says that the radii of D_d are proportional to $\sqrt{d}$.

For arbitrary sequences $(D_d)_{d\in\mathbb{N}}$, we prove that numerical integration suffers from the *curse of dimensionality* for certain classes of smooth functions with suitable bounds on the Lipschitz constants of directional or partial derivatives that may depend on d. The curse of dimensionality means that the minimal number of function evaluations is exponentially large in d. The Lipschitz constants are always defined with respect to the Euclidean distance. This paper is a continuation of our paper [6] with the following new results.

- We provide nontrivial volume estimates, see Theorems 2.1 and 2.3. We prove that the volume of a neighborhood of the convex hull of n arbitrary points is exponentially small in d.
- We obtain matching lower and upper bounds for Lipschitz functions, see Theorem 3.1. We prove that if the radii of D_d are proportional to $\sqrt{d}$ then the curse holds iff $\limsup_{d\to\infty} L_{0,d}\sqrt{d} > 0$, where $L_{0,d}$ is the Lipschitz constant of functions.
- We obtain matching lower and upper bounds for functions with a Lipschitz gradient, see Theorem 4.1. We prove that if the radii of convex D_d are proportional to $\sqrt{d}$ then the curse holds iff $\limsup_{d\to\infty} L_{0,d}\sqrt{d} > 0$ and $\limsup_{d\to\infty} L_{1,d}\, d > 0$, where $L_{1,d}$ is the Lipschitz constant of first directional derivatives of functions.
- We provide lower and upper bounds for functions with higher smoothness $k > 1$, see Theorem 5.1. Our lower bounds are sometimes better than those presented in [6], whereas the upper bounds are new. Unfortunately, our lower and upper bounds do not always match. We prove that if the radii of D_d are proportional to $\sqrt{d}$ then the curse holds if $\limsup_{d\to\infty} L_{0,d}\sqrt{d} > 0$ and $\limsup_{d\to\infty} L_{j,d}\, d > 0$ for all $j = 1, \ldots, k$, where $L_{j,d}$ is the Lipschitz constant of jth directional derivatives of functions. On the other hand, if $\lim_{d\to\infty} L_{j,d} d^{(j+1)/2} = 0$ for some $j \in \{0, 1, \ldots, k\}$ then the curse does not hold. Hence, our bounds match only if $j \in \{0, 1\}$.
- We obtain results for C^∞ functions, see Theorems 6.1 and 7.1. In particular, in this case we also study quasi-polynomial, weak and uniform weak tractability. Quasi-polynomial tractability means that the logarithm of the minimal number of function values that are needed to guarantee an error $\varepsilon > 0$ is bounded proportionally to $(1+\ln\, d)(1+\ln\, \varepsilon^{-1})$, whereas weak tractability means that this number of function values is not exponential in d and ε^{-1}, and uniform weak tractability means that it is not exponential in any positive power of d and ε^{-1}. In particular, we prove that weak tractability holds if all directional derivatives are bounded by one, see Corollary 6.5. It is known that strong polynomial tractability does not hold, i.e., the minimal number of function values cannot be bounded by a polynomial in ε^{-1} independently of d. It is not known if, in particular, we have quasi-polynomial tractability in this case. It is also open if weak tractability holds for the larger class of all partial derivatives bounded by one, see Open Problem 2 of [8].

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